

Homogeneous Structures

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1-Homogeneous

Definition A structure X is 1-homogeneous if X ‘looks the same’ from the perspective of any of its points.

Example 1

Consider a network X modelling some people: The nodes will be the individual people and two people are connected if they are friends. If you pick one person, then his perspective of the network will be, for example, “There are 50 people with whom I am friends, and 49 with whom I am not friends”. Note: whether two other people are friends or not is not a part of that person’s perspective.

In this situation, X is 1-homogeneous if everybody has the same number of friends.

Example 2

The same network as in Example 1, but in a more specific circumstance: X will model the Jets and Sharks. To simplify matters (and warp the story), I’m going to assume two people are friends if and only if they are in the same gang. Since there are 18 Jets and 19 Sharks, we get differing perspectives: a Jet’s perspective (‘There are 17 friends’) will be different from a Shark’s perspective (‘There are 18 friends’).

From the definition I gave, this would mean X is not 1-homogeneous. However, it seems unreasonable that two people who are in different groups should have the same perspective on the whole structure; in a sense, it undermines the fact there are two groups. In other words, 1-homogeneity ought to take into account the fact that two points might be inherently different, so we adjust the definition:

Definition A structure X is 1-homogeneous if for any two points x, y with the same ‘internal properties’, X looks the same from x ’s and y ’s perspective.

I say ‘internal’ properties because, for example, the number of friends a person has could be classed as one of that person’s properties.

Example 3

Lets make the model a bit more accurate: We make Tony and Maria friends. So now, X is not 1-homogeneous, since Tony’s perspective differs from all the other Jets’ perspectives (he now has one more friend than they do).

Example 4

Lets make the question of whether X is 1-homogeneous or not more interesting: imagine Maria told Tony to stop being friends with Riff, and that Tony told Maria to stop being friends with Bernardo (but, without explicitly telling Riff and Bernardo, so these two people’s perspective doesn’t change). So now, all the Jet’s will say they have 17 friends, and all the Shark’s will say they have 18 friends.

So we’re now back in the situation with Example 2, so X is 1-homogeneous. Except that’s not the case. I’m sure you can guess why: although Tony has the same number of

friends as any other Jet, one of those friends is a Shark, so his perspective is in fact different to other Jets. Similarly for Maria.

Example 5

Now onto some mathematical examples. The structure will be numbers and what we are interested in is how they are ordered (i.e. whether one number is bigger or smaller than the other).

First take the positive integers. This is not 1-homogeneous: e.g. 5 will see that there are four numbers smaller than it, whereas 100 will see there are ninety-nine numbers below it.

Now, if you take all the integers, this is 1-homogeneous: given any number, there are infinitely many numbers below and above it.

2-homogeneous

X is 2-homogeneous if given two pairs of points (which have the same internal properties), the perspective on X of one pair is the same as of the other pair.

Example 5

The set of all integers is not 2-homogeneous: e.g. 0 and 5 can say ‘there are four numbers between us’ whereas 3 and 10 can say ‘there are 6 numbers between us.’

Example 1

X is the network of friends. This might get a bit confusing. The internal properties of two people in this situation is ‘we are friends’ or, ‘we are not friends’. The perspective of two people would be the total number of friends they have AND the number of friends they have in common. So, X being 2-homogeneous says that if you take two pairs of friend, the perspectives of each pair is the same, and, if you take two pairs of non-friends, the perspective of each pair is the same.

Example 4

This gets even more confusing, since the ‘internal properties’ of a pair of people can vary and you have to check each case separately. You would have: two sharks who are friends, two jets who are friends, two sharks who are not friends, one shark and one jet who are not friends, one shark and one jet who are friends,...

n-homogeneous

X is n-homogeneous if given two groups of n points (which have the same internal properties), the perspective on X of one group is the same as the perspective of the other group.

X is homogeneous if it is n -homogeneous for all n .

Example: The rational numbers are homogeneous.

End

On re-reading this, I realise some of the things I said aren’t actually accurate (“Whether two other people are friends’ are not is not a part of the perspective”), but, to be really precise about the meaning of ‘perspective of a point’ requires some technical notions which would take a while to go into. Nevertheless, hopefully you have some sort of idea what a homogeneous structure is now.