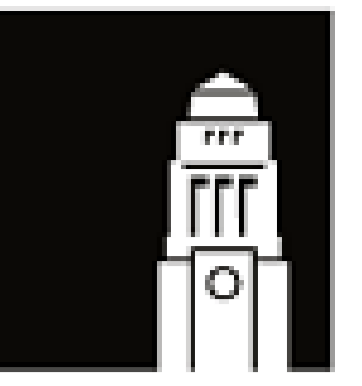


Reducts and Thomas' Conjecture

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Preliminaries

Definition. Let \mathcal{M} be a structure in a language L . A relation P is definable if $\exists \phi(\bar{x}) \in L$ s.t.

$$P = \{\bar{a} \in M : \mathcal{M} \models \phi(\bar{a})\}.$$

Definition. Let \mathcal{M} be a structure. A structure \mathcal{N} is a reduct of \mathcal{M} if \mathcal{N} has the same domain as \mathcal{M} and all definable relations in \mathcal{N} are definable in \mathcal{M} .

Intuition. \mathcal{N} is a reduct of \mathcal{M} if \mathcal{N} is a less detailed version of \mathcal{M} , or, if \mathcal{N} contains less information than \mathcal{M} .

General Question. Given a structure \mathcal{M} , what are its reducts?

Remark. If two reducts $\mathcal{N}_1, \mathcal{N}_2$ of \mathcal{M} are reducts of each other (i.e. inter-definable), they are considered to be the same reduct of \mathcal{M} . Intuitively they contain the same information.

Fact. The reducts of a structure \mathcal{M} form a lattice. For example, the join of two reducts \mathcal{N}_1 and \mathcal{N}_2 is the structure whose relations are those definable in both \mathcal{N}_1 and \mathcal{N}_2 . Intuitively, the join contains the information common to both structures.

Examples

The following definable relations each determine a reduct of $(\mathbb{Q}, <)$:

$$<_w(a, b; x, y) := a < b \leftrightarrow x < y$$

$$\text{cyc}(x, y, z) := x < y < z$$

$$\vee y < z < x$$

$$\vee z < x < y.$$

$$\text{cyc}_w(a, b, c; x, y, z) := \text{cyc}(a, b, c) \\ \leftrightarrow \text{cyc}(x, y, z)$$

(The 'w' abbreviates 'weakened')

Theorem. (Cameron, [1]) The reducts of $(\mathbb{Q}, <)$ are $(\mathbb{Q}, <)$, $(\mathbb{Q}, <_w)$, (\mathbb{Q}, cyc) , $(\mathbb{Q}, \text{cyc}_w)$ and $(\mathbb{Q}, =)$.

Similar theorems have been proved for other structures, for example:

– $(\mathbb{Q}, <, 0)$ has 116 reducts [2]

– The random graph has 5 reducts [3]

– The random k -hypergraph has $2^k + 1$ reducts, for $k \geq 2$ [4]

Thomas' Conjecture

Based on these results, Thomas made a conjecture in his 1996 paper:

Conjecture. If \mathcal{M} is a countable \aleph_0 -categorical structure with quantifier elimination in a finite relational language, then \mathcal{M} has finitely many reducts.

Correspondence with closed groups

There is a central correspondence between reducts and closed subgroups of $\text{Sym}(M)$ - any proof of Thomas' conjecture will undoubtedly use it.

(The topology on $\text{Sym}(M)$ is the subspace topology of the product topology on M^M .)

Fact. For any reduct \mathcal{N} of \mathcal{M} , $\text{Aut}(\mathcal{N})$ is a closed subgroup of $\text{Sym}(M)$ containing $\text{Aut}(\mathcal{M})$.

Fact. If \mathcal{M} is \aleph_0 -categorical, then $\mathcal{N} \mapsto \text{Aut}(\mathcal{N})$ is a lattice isomorphism from the reducts of \mathcal{M} to the closed subgroups of $\text{Sym}(M)$ containing $\text{Aut}(\mathcal{M})$.

Notation. For $F \subseteq \text{Sym}(M)$, let $\langle F \rangle$ be the smallest closed group containing F .

The correspondence for $(\mathbb{Q}, <)$

Let $\leftrightarrow: \mathbb{Q} \rightarrow \mathbb{Q}$ be $q \mapsto -q$.

Let $\circ: \mathbb{Q} \rightarrow \mathbb{Q}$ map (π, ∞) onto $(-\infty, \pi)$, and, $(-\infty, \pi)$ onto (π, ∞) order preservingly. Then:

$$(\mathbb{Q}, <) \mapsto \text{Aut}(\mathbb{Q})$$

$$(\mathbb{Q}, <_w) \mapsto \langle \text{Aut}(\mathbb{Q}) \cup \{\leftrightarrow\} \rangle$$

$$(\mathbb{Q}, \text{cyc}) \mapsto \langle \text{Aut}(\mathbb{Q}) \cup \{\circ\} \rangle$$

$$(\mathbb{Q}, \text{cyc}_w) \mapsto \langle \text{Aut}(\mathbb{Q}) \cup \{\leftrightarrow, \circ\} \rangle$$

$$(\mathbb{Q}, =) \mapsto \text{Sym}(\mathbb{Q})$$

The Generic Directed Graph

My focus is in determining the reducts of the generic digraph. This structure can be defined randomly: Let the domain be \mathbb{N} . For $i < j$, select one of three options with equal probability: edge from i to j , or, edge from j to i , or, no edge at all.

I am using a strategy developed by Bodirsky, Pinsker and Pongrácz: By adding a linear order, Ramsey theory provides, to each reduct, an associated 'nice' function. It suffices to study these 'nice' functions, which boils down to finite combinatorics.

References

[1] P.J. Cameron, *Transitivity of permutation groups on unordered sets*, *Mathematische Zeitschrift*, **148** (1976), 127-139.

[2] M. Junker and M. Ziegler, *The 116 reducts of $(\mathbb{Q}, <, 0)$* , *Journal of Symbolic Logic*, **73**, (2008), 861-884.

[3] S. Thomas, *Reducts of the random graph*, *Journal of Symbolic Logic*, **56** (1991), 176-181.

[4] S. Thomas, *Reducts of random hypergraphs*, *Annals of Pure and Applied Logic*, **80** (1996), 165-193.