Reducts and Thomas' Conjecture

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Preliminaries

Definition. Let $\mathcal{M}$ be a structure in a language L. A relation P is <u>definable</u> if $\exists \phi(\bar{x}) \in L$ s.t. $P = \{\bar{a} \in M : \mathcal{M} \models \phi(\bar{a})\}$.

Definition. Let $\mathcal{M}$ be a structure. A structure $\mathcal{N}$ is a <u>reduct</u> of $\mathcal{M}$ if $\mathcal{N}$ has the same domain as $\mathcal{M}$ and all definable relations in $\mathcal{N}$ are definable in $\mathcal{M}$.

Intuition. $\mathcal{N}$ is a reduct of $\mathcal{M}$ if $\mathcal{N}$ is a less detailed version of $\mathcal{M}$, or, if $\mathcal{N}$ contains less information than $\mathcal{M}$.

General Question. Given a structure $\mathcal{M}$, what are its reducts?

Remark. If two reducts $\mathcal{N}_1, \mathcal{N}_2$ of $\mathcal{M}$ are reducts of each other (i.e. inter-definable), they are considered to be the same reduct of $\mathcal{M}$. Intuitively they contain the same information.

Fact. The reducts of a structure $\mathcal{M}$ form a lattice. For example, the join of two reducts $\mathcal{N}_1$ and $\mathcal{N}_2$ is the structure whose relations are those definable in both $\mathcal{N}_1$ and $\mathcal{N}_2$. Intuitively, the join contains the information common to both structures.

Examples

The following definable relations each determine a reduct of $(\mathbb{Q}, <)$:

$$<_{\mathrm{w}}(a,b;x,y) := a < b \leftrightarrow x < y$$

$$\mathrm{cyc}(x,y,z) := x < y < z \vee y < z < x \vee z < x < y.$$

$$\mathrm{cyc}_{\mathrm{w}}(a,b,c;x,y,z) := \mathrm{cyc}(a,b,c) \leftrightarrow \mathrm{cyc}(x,y,z)$$

(The 'w' abbreviates 'weakened')

Theorem. (Cameron, [1]) The reducts of $(\mathbb{Q}, <)$ are $(\mathbb{Q}, <)$, $(\mathbb{Q}, <_{\mathrm{w}})$, $(\mathbb{Q}, \mathrm{cyc})$, $(\mathbb{Q}, \mathrm{cyc}_{\mathrm{w}})$ and $(\mathbb{Q}, =)$.

Similar theorems have been proved for other structures, for example:

–$(\mathbb{Q}, <, 0)$ has 116 reducts [2]
–The random graph has 5 reducts [3]
–The random k-hypergraph has $2^k + 1$ reducts, for $k \geq 2$ [4]

Thomas' Conjecture

Based on these results, Thomas made a conjecture in his 1996 paper:

Conjecture. If $\mathcal{M}$ is a countable $\aleph_0$-categorical structure with quantifier elimination in a finite relational language, then $\mathcal{M}$ has finitely many reducts.

Correspondence with closed groups

There is a central correspondence between reducts and closed subgroups of $\mathrm{Sym}(M)$ - any proof of Thomas' conjecture will undoubtedly use it.

(The topology on $\mathrm{Sym}(M)$ is the subspace topology of the product topology on M^M.)

Fact. For any reduct $\mathcal{N}$ of $\mathcal{M}$, $\mathrm{Aut}(\mathcal{N})$ is a closed subgroup of $\mathrm{Sym}(M)$ containing $\mathrm{Aut}(\mathcal{M})$.

Fact. If $\mathcal{M}$ is $\aleph_0$-categorical, then $\mathcal{N} \mapsto \mathrm{Aut}(\mathcal{N})$ is a lattice isomorphism from the reducts of $\mathcal{M}$ to the closed subgroups of $\mathrm{Sym}(M)$ containing $\mathrm{Aut}(\mathcal{M})$.

Notation. For $F \subseteq \mathrm{Sym}(M)$, let $\langle F \rangle$ be the smallest closed group containing F.

The correspondence for $(\mathbb{Q}, <)$

Let $\leftrightarrow: \mathbb{Q} \to \mathbb{Q}$ be $q \mapsto -q$.
Let $\circlearrowleft: \mathbb{Q} \to \mathbb{Q}$ map (π, ∞) onto $(-\infty, \pi)$, and, $(-\infty, \pi)$ onto (π, ∞) order preservingly. Then:

$(\mathbb{Q}, <) \mapsto \mathrm{Aut}(\mathbb{Q})$
$(\mathbb{Q}, <_{\mathrm{w}}) \mapsto \langle \mathrm{Aut}(\mathbb{Q}) \cup \{\leftrightarrow\} \rangle$
$(\mathbb{Q}, \mathrm{cyc}) \mapsto \langle \mathrm{Aut}(\mathbb{Q}) \cup \{\circlearrowleft\} \rangle$
$(\mathbb{Q}, \mathrm{cyc}_{\mathrm{w}}) \mapsto \langle \mathrm{Aut}(\mathbb{Q}) \cup \{\leftrightarrow, \circlearrowleft\} \rangle$
$(\mathbb{Q}, =) \mapsto \mathrm{Sym}(\mathbb{Q})$

The Generic Directed Graph

My focus is in determining the reducts of the generic digraph. This structure can be defined randomly: Let the domain be $\mathbb{N}$. For $i < j$, select one of three options with equal probability: edge from i to j, or, edge from j to i, or, no edge at all.

I am using a strategy developed by Bodirsky, Pinsker and Pongrácz: By adding a linear order, Ramsey theory provides, to each reduct, an associated 'nice' function. It suffices to study these 'nice' functions, which boils down to finite combinatorics.

References

[1] P.J. Cameron, *Transitivity of permutation groups on unordered sets*, Mathematische Zeitschrift, **148** (1976), 127-139.

[2] M. Junker and M. Ziegler, *The 116 reducts of* $(\mathbb{Q}, <, 0)$, Journal of Symbolic Logic, **73**, (2008), 861-884.

[3] S. Thomas, *Reducts of the random graph*, Journal of Symbolic Logic, **56** (1991), 176-181.

[4] S. Thomas, *Reducts of random hypergraphs*, Annals of Pure and Applied Logic, **80** (1996), 165-193.