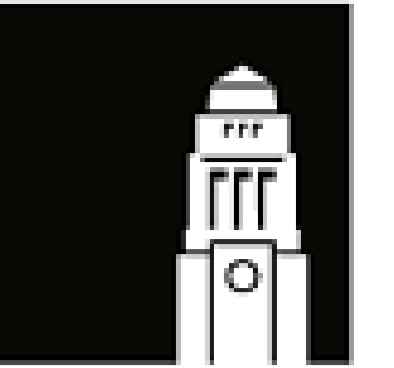


Homogeneous Binary Relational Structures with the same Lattice of Reducts

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Preliminaries

Definition. Let \mathcal{M} be a structure. A structure \mathcal{N} is a *reduct* of \mathcal{M} if \mathcal{N} has the same domain as \mathcal{M} and all \emptyset -definable relations in \mathcal{N} are \emptyset -definable in \mathcal{M} .

Intuition. \mathcal{N} is a reduct of \mathcal{M} if \mathcal{N} is a less detailed version of \mathcal{M} , or, if \mathcal{N} contains less information than \mathcal{M} .

General Question. Given a structure \mathcal{M} , what are its reducts?

Remark 1. If two reducts $\mathcal{N}_1, \mathcal{N}_2$ of \mathcal{M} are reducts of each other, they are considered to be the same reduct of \mathcal{M} ; intuitively they contain the same information.

Remark 2. The reducts of a structure \mathcal{M} form a lattice. For example, the join of two reducts \mathcal{N}_1 and \mathcal{N}_2 is the structure whose relations are those \emptyset -definable in both \mathcal{N}_1 and \mathcal{N}_2 .

A Familiar Structure: $(\mathbb{Q}, <)$

These properties of \mathbb{Q} provide some intuition for the later structures.

- $(\mathbb{Q}, <)$ is \aleph_0 -categorical.
- $(\mathbb{Q}, <)$ embeds all linear orders.
- $(\mathbb{Q}, <)$ is homogeneous: Any $\text{iso}^m f : A \rightarrow B$, $A, B \subset \mathbb{Q}$ finite, can be extended to an auto^m of \mathbb{Q} .
- Let $p(x)$ be a 1-type over a finite parameter set a_1, \dots, a_n . Let $A = \{a \in \mathbb{Q} : a \models p(x)\}$. Then $A = \{a_i\}$ for some i , or, $A \cong \mathbb{Q}$.

Some relations on $(\mathbb{Q}, <)$

We define three relations:

$$<_w(a, b; x, y) := a < b \leftrightarrow x < y$$

$$\text{cyc}(x, y, z) := x < y < z$$

$$\vee y < z < x$$

$$\vee z < x < y.$$

$$\text{cyc}_w(a, b, c; x, y, z) := \text{cyc}(a, b, c) \\ \leftrightarrow \text{cyc}(x, y, z)$$

(‘w’ abbreviates ‘weakened’.)

Reducts of $(\mathbb{Q}, <)$

Theorem. ([Cam76]) The reducts of $(\mathbb{Q}, <)$ are: $(\mathbb{Q}, <)$, $(\mathbb{Q}, <_w)$, (\mathbb{Q}, cyc) , $(\mathbb{Q}, \text{cyc}_w)$ and $(\mathbb{Q}, =)$.

Three other structures

The following structures have the same lattice of reducts as $(\mathbb{Q}, <)$:

- The random graph Γ , [Tho91]
- The random tournament, [Ben97]
- The generic partial order, [PPP⁺11]

(These can be defined as satisfying the earlier properties of \mathbb{Q} but with ‘linear order’ changed appropriately.)

Surprisingly, the reducts are defined in the same way: the original binary relation, its ‘weakened version’, a ‘cyclic’ relation, its ‘weakened version’ and the trivial structure.

Question. Is this just a coincidence? Are there other homogeneous binary structures with the same pattern of reducts?