Homogeneous Binary Relational Structures with the same Lattice of Reducts

Lovkush Agarwal The University of Leeds

Preliminaries

Definition. Let $\mathcal{M}$ be a structure. A structure $\mathcal{N}$ is a *reduct* of $\mathcal{M}$ if $\mathcal{N}$ has the same domain as $\mathcal{M}$ and all $\emptyset$-definable relations in $\mathcal{N}$ are $\emptyset$-definable in $\mathcal{M}$.

Intuition. $\mathcal{N}$ is a reduct of $\mathcal{M}$ if $\mathcal{N}$ is a less detailed version of $\mathcal{M}$, or, if $\mathcal{N}$ contains less information than $\mathcal{M}$.

General Question. Given a structure $\mathcal{M}$, what are its reducts?

Remark 1. If two reducts $\mathcal{N}_1, \mathcal{N}_2$ of $\mathcal{M}$ are reducts of each other, they are considered to be the same reduct of $\mathcal{M}$; intuitively they contain the same information.

Remark 2. The reducts of a structure $\mathcal{M}$ form a lattice. For example, the join of two reducts $\mathcal{N}_1$ and $\mathcal{N}_2$ is the structure whose relations are those $\emptyset$-definable in both $\mathcal{N}_1$ and $\mathcal{N}_2$.

A Familiar Structure: $(\mathbb{Q}, <)$

These properties of $\mathbb{Q}$ provide some intuition for the later structures.

–$(\mathbb{Q}, <)$ is $\aleph_0$-categorical.

–$(\mathbb{Q}, <)$ embeds all linear orders.

–$(\mathbb{Q}, <)$ is homogeneous: Any iso^{m} $f : A \to B$, $A, B \subset \mathbb{Q}$ finite, can be extended to an auto^{m} of $\mathbb{Q}$.

–Let $p(x)$ be a 1-type over a finite parameter set $a_1, \ldots, a_n$. Let $A = \{a \in \mathbb{Q} : a \models p(x)\}$. Then $A = \{a_i\}$ for some i, or, $A \cong \mathbb{Q}$.

Some relations on $(\mathbb{Q}, <)$

We define three relations:

$$<_{\mathrm{w}}(a,b;x,y) := a < b \leftrightarrow x < y$$

$$\mathrm{cyc}(x,y,z) := x < y < z \vee y < z < x \vee z < x < y.$$

$$\mathrm{cyc}_{\mathrm{w}}(a,b,c;x,y,z) := \mathrm{cyc}(a,b,c) \leftrightarrow \mathrm{cyc}(x,y,z)$$

('w' abbreviates 'weakened'.)

Reducts of $(\mathbb{Q}, <)$

Theorem. ([Cam76]) The reducts of $(\mathbb{Q}, <)$ are: $(\mathbb{Q}, <)$, $(\mathbb{Q}, <_{\mathrm{w}})$, $(\mathbb{Q}, \mathrm{cyc})$, $(\mathbb{Q}, \mathrm{cyc}_{\mathrm{w}})$ and $(\mathbb{Q}, =)$.

Three other structures

The following structures have the same lattice of reducts as $(\mathbb{Q}, <)$:

–The random graph Γ, [Tho91]

–The random tournament, [Ben97]

–The generic partial order, [PPP+11]

(These can be defined as satisfying the earlier properties of $\mathbb{Q}$ but with 'linear order' changed appropriately.)

Surprisingly, the reducts are defined in the same way: the original binary relation, its 'weakened version', a 'cyclic' relation, its 'weakened version' and the trivial structure.

Question. Is this just a coincidence? Are there other homogeneous binary structures with the same pattern of reducts?