Wellquasiorders and Betterquasiorders

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1 Introduction

A quasi-order is a set[1] with a binary, transitive, reflexive relation; it is best visualised as a partial order whose points are replaced by a set of 'equivalent' points. This may not seem natural but in the course of this essay it will become evident that quasi-orders are not only natural, but the correct notion. A well-quasi-order (WQO) is a quasi-order which contains no infinite antichains and no infinite (strictly) descending sequences. This notion was first mentioned in the late 30's by Vázsonyi, who conjectured that the set of finite trees with the embeddability relation is WQO, and later by Erdös [Erd49], who asked whether $A \subset \mathbb{N}$ is a WQO implies that the multiplicative closure of A is a WQO, where $\mathbb{N}$ is ordered by divisibility. Terminology for this notion was introduced by Higman [Hig52] and Erdös and Rado [ER52], though the term 'well-quasi-order' was first used by Kruskal in his PhD thesis (1954) and [Kru60], in which he resolved (positively) Vázsonyi's conjecture.

A main goal of WQO theory is determining which naturally occuring quasi-orders are WQO. An example is the class of linear order types (the type of a linear order L, $\mathrm{tp}(L)$, is the isomorphism class of L) with the embeddability relation (if L and M are linear orders then $\mathrm{tp}(L) \leq \mathrm{tp}(M)$ if L is isomorphic to a subset of M). The Dushnik-Miller [DM40] and Sierpinski [Sie50] constructions provide an example of an infinite descending sequence of linear orders and of an infinite antichain, so the class of linear order types is not WQO. The question arises as to which subclasses of linear types are WQO. Fraïssé [Fra48] conjectured that the class of countable types is WQO. He later extended his conjecture to the class of all 'scattered' types, where a linear order is scattered if you cannot embed $\mathbb{Q}$ into it.

The focus of this essay is to present Laver's [Lav71] proof of Fraïssé's conjecture. We mention now that Laver in fact proves that the larger class, $\mathcal{M}$, of *σ-scattered* types is WQO, where L is σ-scattered if it can be written as a countable union of scattered linear orders. We briefly describe the content of the remaining chapters to anticipate the core structure of the proof.

In Chapter 2, we describe WQOs and better-quasi-orders (BQO), not only to establish the results

[1] Or class

needed to prove the main result of Laver's, but also to provide a solid grounding in WQOs and BQOs [1]. BQOs, invented by Nash-Williams [NW65a], are 'stronger versions' of WQOs which have *very* nice properties. In particular, if Q is a BQO, then many structures built from Q are also BQOs. This preservation property allows certain inductive proofs to follow through, which do not for WQOs. Laver's proof is one such example and as a result it turns out we actually prove the stronger result that $\mathcal{M}$ is BQO.

In Chapter 3, we develop some theory on linear orders. We start by reviewing Hausdorff's [Hau1908] recursive characterisation of the class of scattered linear orders, and then generalise this to obtain a recursive characterisation of $\mathcal{M}$. Laver's main result is proved by induction on this characterisation of $\mathcal{M}$.

In Chapter 4, we prove the main result of Laver's paper. For the induction to work, we actually need to add another level of generality, namely to consider Q-orders, which are linear orders whose elements are 'labelled' by elements of Q. Writing $Q^{\mathcal{M}}$ for the class of σ-scattered Q-types, the main theorem of the paper is:

Theorem Q BQO $\Rightarrow Q^{\mathcal{M}}$ BQO.

This is far more general then the original task of proving Fraïssé's conjecture, which is an immediate corollary by considering $Q = \{x\}$, the one element quasi-order.

In Chapter 5, we end with a few comments on Laver's paper and by stating a few related results, which demonstrate the versatility of the ideas in Laver's paper.

Conventions We write On to denote the class of ordinals, and we equate an ordinal with the set of all ordinals smaller than it. We write Card to denote the class of cardinals, and RC to denote the class of regular cardinals. Throughout the paper we assume choice without further concern. In particular, we identify the cardinal κ with the least ordinal whose cardinality is κ.

[1] The title of the essay is *Wellquasiorders and Betterquasiorders* after all!

2 WQOs and BQOs

The purpose of this chapter is to provide the necessary background on the theory of WQOs and BQOs, and also to provide perspective for the main theorem of the essay. The material is based on [Mil85], [For??] and [Lav71]. A few of the proofs given will be by own, but will certainly not contain any original ideas. On the other hand, some of the results of the chapter will be stated without proof, because there is simply too much content.

Definition $\langle Q, \leq_Q \rangle$ is a *quasi-order* iff$_{def}$ $\leq_Q$ is a binary, reflexive ($\forall q \in Q, q \leq q$), transitive ($q \leq r \leq s \Rightarrow q \leq s$) relation on a class Q. If $\leq_Q$ is also anti-symmetric ($q \leq r \leq q \Rightarrow q = r$) then $\langle Q, \leq_Q \rangle$ is a *partial order.*

Definition Two quasi-orders $\langle Q, \leq_Q \rangle$, $\langle Q', \leq_{Q'} \rangle$ are *isomorphic* iff$_{def}$there exists an isomorphism $f : Q \to Q'$, i.e. a bijection $f : Q \to Q'$ such that $\forall q, r \in Q, q \leq_Q r \Leftrightarrow f(q) \leq_{Q'} f(r)$.

Conventions: We often abbreviate $\langle Q, \leq_Q \rangle$ to Q and use $\leq$ instead of $\leq_Q$ when no confusion arises. We use QO as an abbreviation for 'quasi-order' or 'quasi-ordered'. Similary we use PO. Also, we regard two quasi-orders which are isomorphic to be equal, because for our purposes they are indistinguishable.

Examples: $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ with the usual ordering ('less than or equal to') are POs, and hence QOs. Do keep in mind that a set can be ordered in more than one way, e.g. $\mathbb{N}$ ordered by divisibility (so $n \leq m$ iff n divides m) is also a PO.

Remark: The last example highlights the fact that a QO does not have to be total: there can exist $x, y \in Q$ s.t. $x \not\leq y$ and $y \not\leq x$ (in such a situation, we say that x and y are *incomparable.*)

Example of a QO which is not a PO (nor total): The class of all graphs ordered by embeddability (so $G \leq G'$ iff G is isomorphic to a subgraph of G') is a QO, but not anti-symmetric (let G be the complete graph on $\aleph_0$ vertices and G' be the graph obtained by removing one edge from G) nor total (let $G = K_3$ and G' be the graph on four vertices and zero edges).

Definition Let Q be a QO. We say $q, r \in Q$ are *equivalent*, written $q \equiv r$, iff$_{def}$ $q \leq r$ and $r \leq q$.

We write $q < r$ iff$_{def}$ $q \leq r$ and $r \not\leq q$.

Warning: The last definition can trip people up, as they are used to thinking 'if $q \neq r$ and $q \leq r$ then $q < r$', which is not true in general for QOs.

If Q is QO, it is easy to see that $\equiv$ is an equivalence relation. It is also easy to see that $\leq_Q$ induces an ordering on the quotient $Q/\equiv$[1], and further that this makes $Q/\equiv$ into a PO. This means you can think of a QO as a PO whose elements are sets. Visually, imagine taking a Hasse diagram and replacing each dot with a bunch of dots.

Definition A *sequence* (of length ω) in Q is$_{def}$ a function $f : \omega \to Q$. A sequence is *good* iff$_{def}$there are $i < j < \omega$ s.t. $f(i) \leq f(j)$; a sequence is *bad* otherwise. A sequence is *perfect* iff$_{def}$$f(0) \leq f(1) \leq f(2) \leq \ldots$. A sequence f' is a *subsequence* of f iff$_{def}$$f' = f \circ g$ for some strictly increasing $g : \omega \to \omega$.

Definition $A \subseteq Q$ is an *antichain* iff$_{def}$the elements of A are pairwise incomparable. An ω-sequence f is *strictly decreasing* iff$_{def}$for all $i < j < \omega, f(j) < f(i)$.

Definition Let $X \subseteq A \subseteq Q$. X is a *basis* for A iff$_{def}$$A \subseteq \{q \in Q : x \leq q \text{ for some } x \in X\}$.

Lemma 2.1. *Let Q be QO. The following are equivalent:*

(i) Q has no strictly decreasing sequences and does not contain any infinite antichains.

(ii) Every sequence contains a perfect subsequence.

(iii) There are no bad sequences in Q.

(iv) For all $A \subseteq Q$, A has a finite basis.

Proof We will be using Ramsey's Theorem [Ram30]: If $[\omega]^{(2)}$ is finitely coloured, then there exists a monochromatic infinite subset.

[1]Formally, this is dodgy because we could end up having a class which has proper classes as elements. One fixes this by defining $Q/\equiv$ to be the subset of Q formed by picking one element from each equivalence class.

$\neg$(i) $\Rightarrow$ $\neg$(iv): If $A \subseteq q$ is (the range of) a strictly decreasing sequence or is an infinite antichain, then A clearly has no finite basis.

$\neg$(iv) $\Rightarrow$ $\neg$(iii): Let $A \subseteq Q$ not have a finite basis. Then construct a bad sequence $f : \omega \to Q$ recursively: Suppose f has been defined on $\{0, 1, \ldots, n\}$. Since A has no finite basis, $\exists a \in A$ s.t. $f(i) \not\leq a$ for $i = 0, 1, \ldots, n$, and so let $f(n+1) = a$. It is clear by construction that f is bad.

$\neg$(iii) $\Rightarrow$ $\neg$(ii) Any subsequence of a bad ω-sequence is bad.

$\neg$(ii) $\Rightarrow$ $\neg$(i): Let f be a sequence which does not contain a perfect subsequence. Now colour $\{i, j\}$ (WLOG $i < j$): blue if $f(i) \leq f(j)$,

red if $f(j) < f(i)$,

green if $f(i)$ and $f(j)$ are incomparable.

By Ramsey's Theorem, there exists an infinite monochromatic subset. It cannot be blue, as that would give a perfect subsequence. If it is red, then we get a strictly decreasing sequence. If it is green, we get an infinite antichain. $\square$

Definition A QO Q is a well-quasi-order (WQO) iff$_{def}Q$ satisfies the conditions of Lemma 2.1.

The next lemma collects some basic results:

Lemma 2.2. *(i) Any homomorphic image and any subset of a WQO is WQO.*

(ii) If $\leq_1$, $\leq_2$ both quasi-order Q and $q \leq_1 r \Rightarrow q \leq_2 r$, then $\langle Q, \leq_1 \rangle$ WQO $\Rightarrow$ $\langle Q, \leq_2 \rangle$ WQO.

(iii) If $Q_1, Q_2 \subseteq Q$ are both WQO, then $Q_1 \cup Q_2$ is WQO.

(iv) If Q_1, Q_2 are both WQO, then so is $Q_1 \times Q_2$ (where $\langle q, r \rangle \leq \langle q', r' \rangle$ iff$_{def}$ $q \leq_1 q'$ and $r \leq_2 r'$).

Proof (i), (ii) and (iii) are immediate from the definition of WQO. For (iv), we will show that all sequences in $Q_1 \times Q_2$ are good, so let $f = \langle f_1, f_2 \rangle$ be a sequence in Q. f_1 is a sequence in the WQO Q_1, so there exists a perfect subsequence $f_1 \circ g$, so $\forall i < j, f_1(g(i)) \leq_1 f_1(g(j))$. Now $f_2 \circ g$ is a sequence in the WQO Q_2, so there exists $a < b$ s.t. $f_2(g(a)) \leq_2 f_2(g(b))$. But then

$f(g(a)) \leq f(g(b))$ and $g(a) < g(b)$, so f is good. $\square$

Definition Let Q be a QO and $q \in Q$. $Q_q =_{def} \{r \in Q : q \not\leq r\}$.

The next theorem is a direct analogue of ordinal induction:

Theorem 2.3. *(Induction on WQOs) Suppose P is a property where for all WQOs Q, $\big(\forall q \in Q\ (P(Q_q)\big) \Rightarrow P(Q)$. Then $P(Q)$ for all WQOs Q.*

Proof Suppose the conclusion fails and let Q WQO be such that $\neg P(Q)$. We will construct a bad sequence in Q, which contradicts Q WQO and hence giving us the theorem. By the condition on P, there must exist $q_0 \in Q$ s.t. $\neg P(Q_{q_0})$. Let $f(0) = q_0$. Again by the condition on P, there must exist $q_1 \in Q_{q_0}$ s.t. $\neg P((Q_{q_0})_{q_1})$. Let $f(1) = q_1$. Continuing in this fashion gives us a bad sequence, as desired. $\square$

Given a QO Q, there are various constructions, $F(Q)$ say, where the order on Q induces a natural ordering on $F(Q)$, (in [For??], Forster uses the phrase '*lifting* the order from Q to $F(Q)$'). One then asks whether Q WQO implies $F(Q)$ is WQO, for example, Erdös [Erd49] asked if $A \subset \mathbb{N}$ is a WQO implies that the multiplicative closure of A is a WQO, where $\mathbb{N}$ is ordered by divisibility.

The first construcion we consider is the set of sequences in Q:

Definition Let Q be QO. For $\alpha \in \text{On}$, $Q^\alpha =_{def} \{f \text{ s.t. } f : \alpha \to Q\}$, the set of sequences of length α in Q. $Q^{<\alpha} =_{def} \bigcup_{\beta<\alpha} Q^\beta$. Lastly, $Q^{\text{On}} =_{def} \bigcup_{\alpha \in \text{On}} Q^\alpha$. The order $\leq_Q$ induces an order $\leq$ on Q^{On} as follows: $(f : \alpha \to Q) \leq (f' : \alpha' \to Q)$ iff$_{def}$ $\exists g : \alpha \to \alpha'$ s.t. $\forall \beta \in \alpha, f(\beta) \leq_Q f'(g(\beta))$.

Remark: In words, $f < f'$ iff$_{def}$there is a subsequence in f' of length α which is pointwise 'greater than' f. In [For??], Forster uses the visual phrase 'f *stretches* into f'' if $f \leq f'$.

Lemma 2.4. *Q WQO $\Rightarrow Q^{<\omega}$ WQO.*

Proof Suppose the conclusion is false, so there exists a bad sequence in $Q^{<\omega}$. Choose a bad sequence such that its first term, f_0 say, has minimal length. Then from all the bad sequences starting with f_0, choose one such that its second term, f_1 say, has minimal length. Continue so that we get a minimal bad sequence $f(0), f(1), f(2), \ldots$ in the sense that if $f(0), \ldots, f(i-1), g(i), g(i+1), \ldots$ is bad, then the length of $g(i)$ is at least as big as the length of $f(i)$.

Each $f(i)$ is non-empty, otherwise the sequence would not be bad, so let $q(i)$ be the last term of each $f(i)$ and let $g(i)$ be obtained by removing the last term from $f(i)$. Since Q is WQO, there exists a perfect subsequence of the q's, WLOG, $q(i), q(i+1), q(i+2), \ldots$, for some $i < \omega$.

Now consider the sequence $f(0), \ldots, f(i-1), g(i), g(i+1), \ldots$. By minimality of the f's, this sequence is good. Also, $g(j') \leq f(j')$ for any j', so by the badness of the f's, we do not have $f(j) \leq g(j')$ for any $j < i \leq j'$. Hence, there exists $j < k$ s.t. $g(j) \leq g(k)$. But we also have $q(j) \leq q(k)$, so $f(j) \leq f(k)$, contradicting $f(0), f(1), \ldots$ bad. □

Remark: The idea of constructing a minimal bad sequence is very important. An analogue of this is constructed in BQO theory and proves to be very fruitful.

Write $\mathrm{Fin}(Q^{(-)})$ for the sequences of finite range (and of appropriate length) in Q. Rado [Rad54] extended Lemma 2.4 to $\mathrm{Fin}(Q^{<\omega^3})$, and Erdös and Rado [ER59] extended it further to $\mathrm{Fin}(Q^{<\omega^\omega})$. Rado [Rad54] conjectured that Q WQO $\Rightarrow$ $\mathrm{Fin}(Q^{\mathrm{On}})$ WQO, and this was proved by Nash-Williams [NW65a].

We next discuss Vázsonyi's conjecture, that the set of finite trees is WQO by embeddability, where a tree is a connected, acyclic graph. As is common in mathematics, it turns out to be easier to the prove the result for a more general construction. In [Kru60], Kruskal defines a labelled structured tree over X to be a tree T where:

(i) a particular vertex is selected as the root of the tree,

(ii) all the edges are directed so that they point away from this root,

(iii) for each $v \in T$, the set of edges whose initial vertex is v is linear ordered, and

(iv) each $v \in T$ is labelled by an element from a set X.

He then proves the 'Tree Theorem' which states that X WQO $\Rightarrow$ the set of finite labelled structured trees over X is WQO. This proves Vázsonyi's conjecture by taking X to be the one element order.

Nash-Williams [NW63] found a much simpler proof only requiring a tree T to have the additional properties (i) and (ii). Furthermore, the proof can be easily modified to prove the full Tree Theorem. Nash-Williams' proof has since then been refined, mainly by defining trees in a more natural (and general) way.

Definition $\langle T, \leq\rangle$ is a *tree* iff$_{def}\langle T, \leq\rangle$ is a PO s.t. $\forall x \in T, \{y : y \leq x\}$ is well-ordered, and, $\exists\rho(T) \in T$ s.t. $\rho(T) \leq x, \forall x \in T$. $\rho(T)$ is called the *root* of the tree. y is a *successor* of x iff$_{def} x < y$. y is an *immediate successor* of x if $x < y$ and there is no $z \in T$ s.t. $x < z < y$. $S(x)$ is$_{def}$ the set of immediate successors of x. The branch at x, br(x), is$_{def}$ the subtree $\{y : x \leq y\}$ with x as its root.

Definition Let τ be the set of trees which have no paths of length $> w$. For $x, y \in T \in \tau$, $x \wedge y =_{def}$ the glb of x and y (which exists since T has no paths of length $> w$). We quasi-order τ as follows: for $T, T' \in \tau$, $T \leq T'$ iff$_{def} \exists f : T \to T'$ injective and s.t. for all $x, y \in T', f(x) \wedge f(y) = f(x \wedge y)$.

Remark: $f(x) \wedge f(y) = f(x \wedge y)$ implies that f is order-preserving and that if y, z are distinct immediate successors of x, then $f(x)$ and $f(y)$ are distinct successors of $f(x)$. As with embeddability for sequences, you can visualise $T \leq T'$ by thinking 'T stretches into T''.

Lemma 2.5. *τ', the set of finite trees in τ, is WQO.*

Proof We omit the proof, other than to say that the proof has the same structure as in Lemma 2.4: one constructs a minimal bad sequence of trees $T_0, \ldots$ and in place of $g(i)$ and $q(i)$, you consider $\{\text{br}(x) : x \text{ is an immediate succesor of } \rho(T_i)\}$ and $\rho(T_i)$. □

We now go on to describe the natural orderings induced on the powerset of a quasi-order:

Definition Let Q be a QO. Then we quasi-order $\mathcal{P}(Q)$ as follows: $A \leq_m B$ iff$_{def}$ $\exists f : A \to B$ s.t. $\forall a \in A, a \leq_Q f(a)$. If in addition f is injective, then $A \leq_l B$.

Notice that since $\mathcal{P}(Q)$ is QO, we can repeat the construction to get a quasi-order on $\mathcal{P}(\mathcal{P}(Q))$, and repeating again gives a quasi-order on $\mathcal{P}(\mathcal{P}(\mathcal{P}(Q)))$, and so on, defining a quasi-order on $\mathcal{P}^n(Q)$ for every n, where $\mathcal{P}^n(Q)$ is defined in the obvious way.

One can continue this process, defining an order on $\bigcup_{n<\omega}\mathcal{P}^n(Q)$, but the formal definition is not (immediately) intuitive. The main idea is that $\mathcal{P}^n(Q)$ can be naturally embedded into $\mathcal{P}^{n+1}(Q)$ by the function $A \in \mathcal{P}^n(Q) \mapsto \{A\} \in \mathcal{P}^{n+1}(Q)$. So by considering A and $\{A\}$ and $\{\{A\}\}$ and $\{\{\{A\}\}\}$ and ... as all equivalent (for each A), we can think of $\bigcup_{n<\omega}\mathcal{P}^n(Q)$ as a union of a nested sequence of quasi-orders. This is formalised in the next definition, where we actually extend the construction to all ordinals.

Definition $\mathcal{P}^{\mathrm{On}}(Q) = \bigcup_{\alpha\in\mathrm{On}}\mathcal{P}^\alpha(Q)$, where $\mathcal{P}^\alpha(Q)$ is defined recursively: $\mathcal{P}^0(Q) = Q, \mathcal{P}^{\alpha+1}(Q) = \mathcal{P}(\mathcal{P}^\alpha(Q))$ and for limit $\alpha > 0, \mathcal{P}^\alpha(Q) = \bigcup_{\beta<\alpha}\mathcal{P}^\beta(Q)$. To define the ordering on $\mathcal{P}^{\mathrm{On}}(Q)$, let $X, Y \in \mathcal{P}^{\mathrm{On}}(Q)$ and let α, β be minimal such that $X \in \mathcal{P}^\alpha(Q)$ and $Y \in \mathcal{P}^\beta(Q)$. Then, by recursion on (α, β), $X \leq_m Y$ iff$_{def}$:

(i) $\alpha = 0, \beta = 0$ and $X \leq_Q Y$,

(ii) $\alpha = 0, \beta > 0$ and $X \leq_m Y'$ for some $Y' \in Y$,

(iii) $\alpha > 0, \beta > 0$ and $\exists f : X \to Y$ such that $\forall X' \in X, X' \leq_m f(X')$.

$\leq_l$ is defined similarly, but with the added requirement that f is injective in (iii). This construction is very relevant for BQOs. Before going on to discuss BQOs, we give for completeness one final result on WQOs:

Lemma 2.6. *Q WQO $\Rightarrow [Q]^{<\omega}$ (the finite subsets of Q) ordered by $\leq_m$ or $\leq_l$ WQO*

Proof Just as in Lemma 2.4. □

Notice how all our results on WQOs seem to depend on finiteness. One is inclined to ask what happens if we take away the finiteness conditions. In [Rad54], Rado constructed a WQO Q s.t. Q^ω is not WQO, so finiteness is a genuine limitation of WQO theory. Furthermore, Rado's example is canonical in the sense that if Q is WQO but Q^ω is not, then Q contains an isomorphic copy of Rado's example (also proven in [Rad54]). His example is easy to define, though, it is best understood with a diagram[1], so we direct the reader to [Mil85] p.492.

From Rado's example, it is clear that to guarantee $\mathcal{P}(Q)$ or Q^ω is WQO, we need some condition on Q that is stronger than being WQO. Being well-ordered (WO) was such a condition: it is easy to show that Q WO $\Rightarrow \mathcal{P}(Q)$ WQO and in [Mil68] Milner showed that Q WO $\Rightarrow Q^{<\omega^3}$ WQO and further conjectured that Q WO $\Rightarrow Q^{\mathrm{On}}$ WQO. However, most objects of interest are not WO, so these results are of limited consequence.

The breakthrough was made by Nash-Williams with his invention of a better-quasi-order (BQO), which lies strictly in between WQOs and WOs. He invented it as a result of his efforts to show that the class τ is WQO, greatly generalising Kruskal's result. In [NW65a], where he first introduces BQOs, he actually proves the stronger result that τ is BQO. In his following paper [NW65b], he settles Milner's conjecture by proving Q BQO $\Rightarrow Q^{\mathrm{On}}$ BQO.

To be able to define a BQO, we need some preliminary definitions. For convenience, we identify subsets of ω with strictly increasing sequences.

Definition If $t, u \subseteq \omega$ then $t \prec u$ iff$_{def}$ t is an initial segment of u. If $t, u \in [\omega]^{<\omega}$ then $t \lhd u$ iff$_{def}$ $\exists i_1, \ldots, i_m$ s.t. $t = \{i_1, ..., i_k\}$ for some $k \leq m$, and $u = \{i_2, \ldots, i_m\}$.

Definition An infinite $B \subset [\omega]^{<\omega}$ is a *block* iff$_{def}$ $\forall X \in \left[\bigcup B\right]^\omega$ $\exists! b \in B$ s.t. $b \prec X$. If in addition B is a $\subseteq$-antichain then B is a *barrier*.

Examples: $[\omega]^n$ for any $n \in \omega$ is a barrier. If B is a block and $A \in \left[\bigcup B\right]^\omega$, then $\{b \in B : b \subset A\}$ is a block.

Definition Let Q be a QO. Then a Q-*pattern* is$_{def}$ a function $f : B \to Q$ where B is a barrier. A Q-pattern is *good* iff$_{def}$there are $b \lhd b'$ s.t. $f(b) \leq f(b')$; a Q-pattern is *bad* otherwise. A Q-pattern is *perfect* iff$_{def}$ $b \lhd b'$ implies $f(b) \leq f(b')$.

Definition Q is better-quasi-ordered (BQO) iff$_{def}$ there are no bad Q-patterns.

[1]Unfortunately, my LaTeX skills are not yet good enough to handle diagrams.

Nash-Williams said in his paper that he defined a BQO by extrapolating from Rado's example conditions on Q that would ensure $\mathcal{P}^n(Q)$ is WQO, for all $n < \omega$ under the $\leq_m$ ordering. He also claimed without proof that Q BQO iff $\mathcal{P}^\alpha(Q)$ WQO for all $\alpha \in \mathrm{On}$. Laver [Lav71] and Pouzet [Pou72] obtained Q BQO iff $\mathcal{P}^{\omega_1}(Q)$ is WQO. To develop an intuition for BQOs, I highly recommend reading [For??].

The next result due to Galvin and Prikry [GP73] is effectively the analogue of Ramsey's Theorem for BQO theory. The proof is omitted.

Theorem 2.7. *If $Y \subseteq [\omega]^{<\omega}$ then there exists $A \in [\omega]^\omega$ s.t. either $Y \cap [A]^{<\omega} = \emptyset$, or, $\forall X \in [A]^\omega, \exists y \in Y$ s.t. $y \prec X$.*

Corollary 2.8. *If B is a block and $B_1 \cup B_2 = B$, then B_1 or B_2 contains a block C.*

Proof WLOG $\bigcup B = \omega$. Apply Thm 2.7 to $Y = B_1$ to obtain A. Let $C = \{b \in B : b \subset A\}$ so C is a block. Note that $\bigcup C \subseteq A$. If $Y \cap [A]^{<\omega} = \emptyset$, then $B_1 \cap C = \emptyset$ so (and because $\bigcup B = \omega$) $C \subseteq B_2$.

On the other hand, suppose $\forall X \in [A]^\omega, \exists y \in Y$ s.t. $y \prec X$. Now let $c \in C$ and $X \in \left[\bigcup C\right]^\omega$ s.t. $c \prec X$. Since B is a block, if $b \in B$ and $b \prec X$, then $b = c$. But $\bigcup C \subseteq A$, so $X \in [A]^\omega$, so $\exists y \in Y = B_1$ s.t. $y \prec X$, so $c = y \in B_1$. Hence, $C \subseteq B_1$. □

Corollary 2.9. *Every block contains a barrier*

Proof Let B be a block (WLOG $\bigcup B = \omega$) and $Y \subset B$ contain all the $\subseteq$-minimal elements of B, so in particular, Y is a $\subseteq$-antichain. Let A be the set obtained by applying Thm 2.7 to Y, and let $C = \{b \in B : b \subset A\}$; C is a block. Since any $\subseteq$-minimal element of C is in Y, we cannot have $Y \cap [A]^{<\omega} = \emptyset$, hence $\forall X \in [A]^\omega, \exists y \in Y$ s.t. $y \prec X$. As in Cor 2.8, $C \subseteq Y$, so C is a barrier. □

Corollary 2.10. *Every Q-pattern contains a bad or a perfect sub-pattern.*

Proof Let $f : B \to Q$ be a Q-pattern. Let $B' = \{b_1 \cup b_2 : b_1, b_2 \in B, b_1 \lhd b_2\}$. Given $b \in B'$ there are unique $b_1, b_2 \in B$ s.t. $b = b_1 \cup b_2$: b_1 will be the unique $b' \in B$ s.t. $b' \prec b$ (unique since B is a barrier) and b_2 has to equal $b-\min(b)$. Next we claim that B' is a barrier. To prove the claim, let $X \in \left[\bigcup B'\right]^{\omega}$. Then $X, X-\min(X) \in \left[\bigcup B\right]^{\omega}$ so $\exists!\, b_1, b_2 \in B$ s.t. $b_1 \prec X$ and $b_2 \prec X-\min(X)$. But then $b = b_1 \cup b_2$ is the unique $b \in B'$ s.t. $b \prec X$, so B' is a block. If $b \subseteq b'$ are both in B', then $b_2 \subseteq b_2'$ are both in B, contradicting B $\subseteq$-minimal, hence B' is $\subseteq$-minimal, thus proving the claim.

Now write B' as $R \cup S$ where $R = \{b \in B' : f(b_1) \leq f(b_2)\}$ and $S = \{b \in B' : f(b_1) \not\leq f(b_2)\}$. By Cor 2.8 and 2.9, either R or S contains a barrier C. Let $C' = \bigcup_{b \in C}\{b_1, b_2\}$. But then, f restricted to C' is a perfect or a bad sub-pattern of f depending on whether $C \subseteq R$ or S. □

The next theorem collects some basic results on BQOs. These are particularly important for us as we make direct use of them in Chapter 4. In the statement of the next theorem, a * indicates that no proof is provided.

Theorem 2.11. *(i) Q BQO $\Rightarrow Q$ WQO.*

(ii) Q well ordered $\Rightarrow Q$ BQO.

(iii) $Q = Q_1 \cup Q_2$ and Q_1, Q_2 BQO $\Rightarrow Q$ BQO.

(iv) Q_1, Q_2 BQO $\Rightarrow Q = Q_1 \times Q_2$ BQO.

(v) Q BQO $\Rightarrow Q^{<\omega}$ BQO.*

(vi) Q BQO $\Rightarrow \mathcal{P}(Q)$ BQO (under $\leq_m$ or $\leq_l$).

(vii) Q BQO, $f : Q \to Q'$ order preserving and $\forall q \in Q'\ \exists g(q) \in Q$ s.t. $f(g(q)) \equiv q \Rightarrow Q'$ BQO.

Proof (i) By considering the barrier $[\omega]^1$, it is immediate that a bad sequence is a bad Q-pattern.

(ii) Suppose Q is WO and $f : B \to Q$ is any Q-pattern. Then there exists $b \in B$ s.t. $f(b) \leq f(b')$ for all $b' \in B$. Now let $X \in \left[\bigcup B\right]^{\omega}$ be such that $b-\min(b)$ is an initial segment of X. By the definition of a block, there exists $b' \in B$ s.t. b' is an initial segment of X. By definition of a barrier, $b' \not\subseteq b-\min(b)$. Hence we have $b \lhd b'$ and $f(b) \leq f(b')$, so f is good, so Q is BQO.

(iii) Suppose $f : B \to Q$ is a bad Q-pattern. Let $B_1 = f^{-1}(Q_1)$ and $B_2 = f^{-1}(Q_2)$. Then by Cor 2.8 and 2.9, $\exists$ barrier $B' \subseteq B_i$ for some i. But then $f : B' \to Q_i$ is a bad Q_i-pattern.

(iv) Let $f : B \to Q$ be a Q-pattern. Then $f_i = \pi_i \circ f$ is a Q_i-pattern. By Cor 2.10 and Q_1

BQO, $\exists$ sub-barrier B' of B s.t $f_1|_{B'}$ is a perfect Q_1-pattern, so for all $b \lhd b' \in B'$, $f_1(b) \leq_1 f_1(b')$. Since Q_2 is BQO, $f_2|_{B'}$ is a good Q_2-pattern so $\exists\, b \lhd b' \in B'$ s.t. $f_2(b) \leq_2 f_2(b')$. Hence, we have $b \lhd b' \in B$ s.t. $f(b) \leq f(b')$, so f is a good Q-pattern.

(vi) Suppose $f : B \to \mathcal{P}(Q)$ is a bad $\mathcal{P}(Q)$-pattern. As in the proof of Cor 2.10, let $B' = \{b_1 \cup b_2 : b_1, b_2 \in B, b_1 \lhd b_2\}$. We will construct a bad Q-pattern $g : B' \to Q$ as follows. Let $b \in B'$ and let $b_1 \lhd b_2 \in B$ be unique such that $b = b_1 \cup b_2$. Since $f(b_1) \not\leq_m f(b_2)$, there is a $g(b) \in f(b_1)$ s.t. $g(b) \not\leq q$ for all $q \in f(b_2)$. Now if $b \lhd c \in B'$, then $b_2 = c_1$ so $g(b) \not\leq g(c)$, hence, g is bad.

(vii) If $h : B \to Q'$ is bad, then $g \circ h : B \to Q$ is bad. $\square$

(vii) is known as the 'homomorphism property' and (vi) is a distinguishing factor for BQOs. In fact, much more than (vi) is true: Laver [Lav78] showed that Q BQO $\Rightarrow \langle \mathcal{P}^{\mathrm{On}}(Q), \leq_l \rangle$ BQO. (v) is also subsumed by a much bigger result: Nash-Williams [NW68] proved Q BQO $\Rightarrow Q^{\mathrm{On}}$ BQO. These two results are proved by using a generalised version of the 'minimal bad sequence' idea that we used for WQOs.

We end the section by stating a generalisation of Nash-Williams result, τ is BQO, that we need in Chapter 4.

Definition (T, l) is a Q-tree iff$_{def}T$ is a tree and $l : T \to q$. Intuitively, l is labelling the vertices of T with elements of Q. Let τ_Q be the set of Q-trees (T, l) where $T \in \tau$. Quasi-order τ_Q as follows: $(T, l) \leq (T', l')$ iff$_{def}$ $T \leq T'$ by a function f s.t. $l(t) \leq_Q l'(f(t))$ for all $t \in T$.

Theorem 2.12. *(Laver [Lav71]) Q BQO $\Rightarrow \tau_Q$ BQO.*

However, for our purposes, we want a slightly modified embeddability relation on our Q-trees:

Definition $(T, l) \leq_m (T', l')$ iff$_{def}$ there is a strictly increasing $f : T \to T'$ s.t. $l(x) \leq_Q l'(f(x))$ for all $x \in T$. (The difference is that f need not be injective, nor does it have to send distinct immediate successors of x to distinct successors of $f(x)$.)

Corollary 2.13. *Q BQO $\Rightarrow \langle \tau_Q, \leq_m \rangle$ BQO.*

Proof $(T, l) \leq (T', l') \Rightarrow (T, l) \leq_m (T', l')$, so it is immediate from Thm 2.12 and the definition of BQO.

3 Linear Orders

The main goal of this chapter is to give a recursive characterisation of $\mathcal{M}$, the class of σ-scattered linear orders. We start by recalling/introducing the core definitions and constructions.

Definition A *linear order* $\langle L, \leq \rangle$ is a set L equipped with a binary, transitive, anti-symmetric, total relation $\leq$. $\langle L, \leq_L \rangle$ and $\langle M, \leq_M \rangle$ are *isomorphic* if there is a bijection $f : L \to M$ s.t. $a \leq_L b$ iff $f(a) \leq_M f(b)$. The *(order) type* of L, written $\mathrm{tp}(L)$, is the class of linear orders isomorphic to L.

Write LO for the class of all (linear) order types. In this essay L, M, N and ϕ, ψ, θ will range over linear orders and order types respectively.

Definition For $\phi, \psi \in \mathrm{LO}$, $\phi \leq \psi$ iff$_{def}$ $\exists L, M$ and $f : L \to M$ such that $\mathrm{tp}(L) = \phi$, $\mathrm{tp}(M) = \psi$ and f embeds L into M, i.e. f is injective and order preserving.

The fact that $\langle LO, \leq \rangle$ is a quasi-order is immediate from the definition. Note that $\leq$ is not anti-symmetric (consider intervals $[0, 1)$ and $(0, 1]$) nor total (consider $\mathbb{N}$ and $\mathbb{Z} \backslash \mathbb{N}$), so LO is another example of a QO which is not a PO.

Definition The *ordered sum* $\sum_{x \in L} M_x$ is the linear order obtained by replacing each $x \in L$ by M_x. (Formally, $\sum_{x \in L} M_x$ is the set $\{(x, y) : x \in L, y \in M_x\}$ ordered lexicographically). $\sum_{x \in L} \psi_x$ is the type of $\sum_{x \in L} M_x$ where $\mathrm{tp}(M_x) = \psi_x$ for all x. The product $\psi \cdot \phi$ equals$_{def}$ $\sum_{x \in L} \psi_x$ where $\mathrm{tp}(L) = \phi$ and $\psi_x = \psi$ for all x. (Think 'ϕ copies of ψ'.)

Definition The *converse* of $\langle L, \leq \rangle$ is $\langle L, \geq \rangle$, where $x \geq y$ iff$_{def}$ $y \leq x$. The converse of the type ϕ, written ϕ^*, is the type of $\langle L, \geq \rangle$ where $\mathrm{tp}(\langle L, \leq \rangle) = \phi$. Write On^* for the class of converses of well-orders.

Definition We write η for the order type of $\mathbb{Q}$, and say ϕ, resp. L, is *scattered* iff$_{def}$ $\eta \not\leq \phi$, resp. $\mathbb{Q} \not\leq L$. We write $\mathcal{S}$ for the class of all scattered types.

Examples: Any finite linear order is scattered. If $\alpha \in \mathrm{On}$, then α, resp. α^* is scattered since all subsets of α, resp. α^*, have a minimum, resp. maximum, element.

Terminology: We say ψ is a 'ϕ sum' if $\psi = \sum_{x \in L} \psi_x$ where $\text{tp}(L) = \phi$. Furthermore, if ϕ is in On / On* / $\mathcal{S}$ / a subclass $\mathcal{R}$ of LO, we say ψ is a 'well ordered / conversely well ordered / scattered /$\mathcal{R}$ sum'.

The following lemma collects some important basic results:

Lemma 3.1. *(i) $tp(L) = \eta$ iff L is countable, dense and unbounded.*

(ii) If $\eta \leq \sum_{x \in L} tp(M_x)$, then either $\eta \leq tp(L)$ or $\eta \leq tp(M_x)$ for some x. ('A scattered sum of scattered types is scattered.')

(iii) If $\kappa \in RC$ and $\kappa \leq \sum_{x \in L} tp(M_l)$, then either $\kappa \leq tp(L)$ or $\kappa \leq tp(M_x)$ for some x.

(iv) If $\kappa \in RC, \alpha < \kappa, L = \bigcup_{\beta < \alpha} L_\beta$ and $\kappa \leq tp(L)$, then $\kappa <tp(L_\beta)$ for some β.

(v) If $\kappa \in RC, tp(L) \in \mathcal{S}$ and $|L| \geq \kappa$, then κ or $\kappa^* \leq tp(L)$.

Remark: The proof of (v) is given later, as it uses Thm 3.2, which in turn uses (ii) above.

Proof (i) If $\text{tp}(L) = \eta$, then L is countable, dense and unbounded, because $\mathbb{Q}$ is. Conversely, suppose L is countable, dense and unbounded. Let $\{l_0, l_1, ...\}$ and $\{q_0, q_1, ...\}$ be enumerations of L and $\mathbb{Q}$. Then construct an isomorphism $f : L \to \mathbb{Q}$ inductively:

$$\begin{aligned} f(l_0) &= q_0 \\ f(l_n) &= q_k, \text{where } k \text{ is the least so that } f \text{ is order preserving and injective.} \end{aligned}$$

Such a k exists at each stage, since $\mathbb{Q}$ is dense and unbounded, so f is well-defined. f is injective and order-preserving by construction. Suppose f is not surjective; then there is a least k such that q_k is not in the image of f. WLOG, $f(l_i) = q_i$ for all $i < k$, and $q_0 < ... < q_{k-1}$. Now, either $q_i < q_k < q_{i+1}$ for some i, or, $q_i < q_k \forall i < k$, or, $q_k < q_i \forall i < k$. In the first case, since L is dense, there exists $n > k$ s.t. $l_i < l_n < l_j$; choose n to be the least such. But then by definition of f, $f(l_n) = q_k$. Contradiction. Similarly for the other two cases, but you use the fact L is unbounded. Hence, f is indeed an isomorphism from L to $\mathbb{Q}$.

(ii) Suppose f is an embedding from $\mathbb{Q}$ to $\sum_{l \in L} M_l$. Let $A_x = \{q \in \mathbb{Q} : f(q) \in M_x\}$. Split into two cases: whether or not there is an $x \in L$ such that $|A_x| \geq 2$.

If so, let $p, q \in \mathbb{Q}$ distinct and $x \in L$ s.t. $p, q \in A_x$. Since f is order preserving, (and WLOG $p < q$), the interval (p, q) must be a subset of A_x and so $f|_{(p,q)} : (p, q) \to M_x$ is an embedding. Observe that (p, q) is countable, dense and unbounded, so by (i), tp((p, q))=η, so $\eta \leq \text{tp}(M_x)$.

If not, then $\forall x \in L, |A_x| \leq 1$. Then, let $g : \mathbb{Q} \to L$ be the function sending q to the $x \in L$ s.t. $q \in A_x$. Clearly, g is an embedding of $\mathbb{Q}$ into L, so $\eta \leq \text{tp}(L)$.

(iii) Suppose we have an embedding $f : \kappa \to \sum_{x \in L} M_x$. Since κ is isomorphic to any subset A of κ of size κ (this is seen by enumerating A as $A = \{\alpha_\gamma : \gamma < \kappa\}$ where $\alpha_0 < \alpha_1 < \dots$), it suffices to show that for some $A \subseteq \kappa$ of size κ, either $A \leq L$ or $A \leq M_x$ for some x.

Let $A_x = \{\alpha \in \kappa : f(\alpha) \in M_x\}$. Let $L' = \{x \in L : A_x \neq \emptyset\}$. Observe that $x < x'$ iff all elements of A_x are less than all elements of $A_{x'}$, by the definition of an ordered sum together with the fact that f is order preserving. Now, $\kappa = \sum_{x \in L'} A_l$ and because κ is regular, $|A_x| = \kappa$ for some x, or, $|L'| = \kappa$.

If $|A_x| = \kappa$ for some x, then f restricted to A_x is an embedding of A_x (a subset of size κ) into M_x.

If $|L'| = \kappa$, then for each $x \in L'$, let y_x be any member of A_x. Let $Y = \bigcup_{x \in L'} y_x$. We have $|Y| = |L'| = \kappa$. Let $g : Y \to L, g(y_x) = x$. This is an embedding from Y into L (order preserving because of the observation made two paragraphs earlier).

(iv) Suppose we have an embedding $f : \kappa \to \bigcup_{\beta < \alpha} L_\beta$. For each $\beta < \alpha$, let $A_\beta = \{\gamma \in \kappa : f(\gamma) \in L_\beta \text{ and } f(\gamma) \notin L_\delta, \forall \delta < \beta\}$. Then $\bigcup_{\beta < \alpha} A_\beta = \kappa$. Since κ is regular and $\alpha < \kappa$, $|A_\beta| = \kappa$ for some β. Then f restricted to A_β is an embedding of A_β into L_β, so $\eta \leq \text{tp}(L_\beta)$. □

A (neat!) corollary of (i) is that there is an embedding from any countable linear order L into $\mathbb{Q}$. You consider $L' = \sum_{x \in L} \mathbb{Q}_x$, where $\forall x, \mathbb{Q}_x = \mathbb{Q}$. You can embed L into L' (map x to any element of $\mathbb{Q}_x$) and moreover, L' is countable, dense and unbounded, so by (i), L' is isomorphic to $\mathbb{Q}$. Hence, you get an embedding from L to $\mathbb{Q}$.

The next result is Huasdorff's recursive characterisation of $\mathcal{S}$. The proof given is my own, mirroring the proof of Thm 3.8[1] (the recursive characterisation of $\mathcal{M}$).

Theorem 3.2. *$\mathcal{S} = \bigcup_{\alpha \in On} \mathcal{S}_\alpha$ where $\mathcal{S}_\alpha$ is defined recursively: $\mathcal{S}_0 = \{0, 1\}$, and for $\alpha > 0, \mathcal{S}_\alpha = \left\{\phi \in LO\text{: } \phi \text{ is a well ordered or conversely well ordered sum of elements in } \bigcup_{\beta<\alpha} \mathcal{S}_\beta\right\}$.*

Proof Let $T = \bigcup_{\alpha \in \mathrm{On}} \mathcal{S}_\alpha$. We will show that $\mathcal{S} \subseteq T$ and $T \subseteq \mathcal{S}$.

$T \subseteq \mathcal{S}$: We show, by induction on On, that $\mathcal{S}_\alpha \subset \mathcal{S}$ $\forall \alpha \in$ On. 0 and 1 are trivially scattered, so $\mathcal{S}_0 \subseteq \mathcal{S}$. For $\alpha > 0$, $\phi \in \mathcal{S}_\alpha$ means that ϕ is a (conversely) well ordered sum of elements of $\bigcup_{\beta<\alpha} \mathcal{S}_\beta$. In particular, by the inductive hypothesis, ϕ is a scattered sum of scattered types, so by Lemma 3.1 (i), ϕ is scattered, i.e. $\phi \in \mathcal{S}$.

$\mathcal{S} \subseteq T$: First we claim that a T-sum of elements of T is in T. We prove this by induction on α on the proposition: An $\mathcal{S}_\alpha$-sum of elements of T is in T. This is true when $\alpha = 0$, so suppose $\alpha > 0$, $\mathrm{tp}(M) \in \mathcal{S}_\alpha$, and $\phi_x \in T$ for all $x \in M$. By definition, $M = \sum_{\delta \in \gamma} M_\delta$, where $M_\delta \in \bigcup_{\beta<\alpha} \mathcal{S}_\beta$ and $\gamma \in$ On (the case $\gamma \in \mathrm{On}^*$ is symmetric). Then,

$$\begin{aligned} \phi \text{ (say) } = \sum_{x \in M} \phi_x &= \sum_{\delta \in \gamma} \sum_{x \in M_\delta} \phi_x \\ &= \sum_{\delta \in \gamma} \phi'_\delta, \quad \text{where } \phi'_\delta = \sum_{x \in M_\delta} \phi_x. \end{aligned}$$

By the induction hypothesis, $\phi'_\delta \in T$ for all δ. Hence, $\forall \delta\ \phi'_\delta \in \mathcal{S}_{\zeta_\delta}$ for some $\zeta_\delta \in$ On. Hence, $\forall \delta, \phi'_\delta \in \mathcal{S}_\zeta$ where $\zeta = \sup_{\delta \in \gamma} \zeta_\delta$. But then $\phi \in \mathcal{S}_{\zeta+}$, so $\phi \in T$. Thus the claim is true.

Now, let L be a linear order with $\mathrm{tp}(L) \in \mathcal{S}$. Define a relation $\sim$ on L: let $x \sim y$ iff $x = y$, or, $x < y$ and $\mathrm{tp}((x, y)) \in T$, or, $y < x$ and $x \sim y$. It is clear that $\sim$ is an equivalence relation which partitions L into intervals. We claim that if $|x|$ is an equivalence class of $\sim$, then $\mathrm{tp}(|x|) \in T$. To show this, let $\langle x_\delta : \delta \in \gamma \rangle$, resp. $\langle x'_{\delta'} : \delta' \in \gamma' \rangle$ be a strictly increasing, resp. decreasing, unbounded sequence in $|x|$ s.t. $x_0 = x = x'_0$. Then $|x|$ can be written as a $(\gamma'^* + 1 + \gamma)$-sum of intervals:

$$|x| = \left(\sum_{\delta'^* \in \gamma'^*} \left[x'_{\delta'+}, x'_{\delta'}\right) \right) + x + \left(\sum_{\delta^* \in \gamma^*} \left(x_\delta, x_{\delta+}\right] \right)$$

By the definition of $\sim$, all these intervals have type in T, and $(\gamma'^* + 1 + \gamma) \in T$, hence $|x|$ is a T-sum of elements of T, so by the previous claim, $\mathrm{tp}(|x|) \in T$, thus proving this claim.

[1]Though for the reader, the proof of Thm 3.8 will be doing the mirroring.

We now claim that there is in fact only one equivalence class, L, which gives the theorem because, by the second claim, we would get $\mathrm{tp}(L) \in T$. So suppose the claim is false. Let L' be the set of equivalence classes of L, ordered by $|x| \leq |y|$ iff $x \leq y$. For any distinct $|x|, |y| \in L'$, $\mathrm{tp}((|x|, |y|)) \notin T$ $(*)$. Otherwise $\mathrm{tp}(x, y)$ would be a T sum of elements of T, which, by the first claim, contradicts $x \not\sim y$.

On the other hand, L' is scattered because L is scattered. Hence, there exists $|x|, |y|$ distinct such that $(|x|, |y|)$ is empty; otherwise L' would contain a dense, unbounded countable subset, which, by Lemma 3.1 (i), contradicts L' scattered. But such an interval has type 0, which is certainly in T, contradicting $(*)$. □

Proof of Lemma 3.1 (v) (If $\kappa \in RC, \mathrm{tp}(L) \in \mathcal{S}$ and $|L| \geq \kappa$, then κ or $\kappa^* \leq \mathrm{tp}(L)$)

By Thm 3.2, (and taking the contrapositive), this is equivalent to proving that for every α, if $\kappa \in RC, \mathrm{tp}(L) \in \mathcal{S}_\alpha$ and $\kappa, \kappa^* \not\leq \mathrm{tp}(L)$, then, $|L| < \kappa$.

This trivially holds for $\alpha = 0$, so let $\alpha > 0$, $\mathrm{tp}(L) \in \mathcal{S}_\alpha$ and $\kappa \in$ RC be such that $\kappa, \kappa^* \not\leq \mathrm{tp}(L)$. By the definition of $\mathcal{S}_\alpha, L = \sum_{\delta \in \gamma} L_\delta$, where $\mathrm{tp}(L_\delta) \in \bigcup_{\beta < \alpha} \mathcal{S}_\beta$ and WLOG $\gamma \in$ On (the case $\gamma \in \mathrm{On}^*$ is symmetric). Then,

$$
\begin{aligned}
\kappa, \kappa^* \not\leq \ \mathrm{tp}(L) \ &\Rightarrow \ \kappa, \kappa^* \not\leq \ \mathrm{tp}(L_\delta) \ \forall \delta \text{ and } \kappa, \kappa^* \not\leq \gamma \\
&\Rightarrow \ |L_\delta| < \kappa \ \forall \delta \text{ and } |\gamma| < \kappa \text{ (by the induction hypothesis)} \\
&\Rightarrow \ |L| < \kappa \text{ (because } \kappa \text{ is regular), \quad as required.}
\end{aligned}
$$

□

We now define $\mathcal{M}$, the class of σ-scattered order types:

Definition $\phi \in \mathcal{M}$ iff$_{def}$ $\mathrm{tp}(L) = \phi \Rightarrow \exists L_0, L_1, ...$ s.t. $L = \bigcup_{n \in \omega} L_n$ and $\mathrm{tp}(L_n) \in \mathcal{S}$ $\forall n$.

Examples: All scattered types are in $\mathcal{M}$. η is in $\mathcal{M}$. An $\mathcal{M}$-sum of elements of $\mathcal{M}$ is in $\mathcal{M}$ (simple rearranging of unions and sums, and use of the fact that a scattered sum of scattered types is scattered).

Before continuing, we sketch what will follow. For particular pairs of cardinals $\langle \alpha, \beta \rangle$, we construct types $\eta_{\alpha\beta} \in \mathcal{M}$. A key feature is that $\phi \in \mathcal{M}$ iff $\phi \leq \eta_{\alpha\beta}$ for some $\eta_{\alpha\beta}$. Then $\mathcal{D}_{\alpha\beta}$ is defined (it is just $\{\phi : \phi < \eta_{\alpha\beta}\}$) and a recursive construction of $\mathcal{D}_{\alpha\beta}$ is given, which is the recursive characterisation of $\mathcal{M}$ that has been previously mentioned.

The following definitions will seem quite random, but it will all fit into place in the lemmas which follow.

Definition We say $\langle \alpha, \beta \rangle \in \text{Card}\times\text{Card}$ is *admissable* iff$_{def}$ α, β are regular and uncountable, and $\max\{\alpha, \beta\}$ is a succesor cardinal.

Definition We define a type $\sigma_{\alpha\beta}$ for every admissable $\langle \alpha, \beta \rangle$ as follows:
For succesor cardinals $\alpha = \gamma^+$ and $\beta = \delta^+, \sigma_{\alpha\beta} = \gamma^* \cdot \delta$.
For α a limit cardinal (hence $\beta = \delta^+$ and $\alpha < \beta$), $\sigma_{\alpha\beta} = \sum_{x \in M} \phi_x$, where $\text{tp}(M) = \delta, \phi_x < \alpha^*$ for all m, and for all $\alpha' < \alpha^*, \exists x$ s.t. $\phi_x \geq \alpha'$.
For β a limit cardinal, $\sigma_{\alpha\beta} = (\sigma_{\beta\alpha})^*$.

Definition For $\langle \alpha, \beta \rangle$ admissable, $\eta_{\alpha\beta} = \text{tp}(L)$, where $L = \bigcup_{n \in \omega} L_n$ and $L_0 \subset L_1 \subset ...$ are defined recursively: $L_0 = \sigma_{\alpha\beta}$, and L_{n+1} is obtained by inserting a copy of $\sigma_{\alpha\beta}$ into every empty interval in L_n.

Remark: The non-uniqueness of $\sigma_{\alpha\beta}$ in the limit cardinal cases is not important because, as will be shown in Thm 3.6, the resulting $\eta_{\alpha\beta}$ is unique (up to $\equiv$-equivalence).

Remark: It is easy to see that $\eta_{\aleph_1\aleph_1}$ is countable, dense and unbounded, so by Lemma 3.1 (i), $\eta_{\aleph_1\aleph_1} = \eta$. This observation may provide you with some intuition in the following lemmas.

Lemma 3.3. *Let L be a linear order of type $\eta_{\alpha\beta}$ for some (admissable) $\langle \alpha, \beta \rangle$, then:*

(i) $\eta_{\alpha\beta} \in \mathcal{M}$.

(ii) $\alpha^* \not\leq \eta_{\alpha\beta}$ *and* $\beta \not\leq \eta_{\alpha\beta}$.

(iii) $\forall x < y \in L, \alpha' \leq (x, y)$ *for all* ordinals $\alpha' < \alpha^*$ *and* $\beta' \leq (x, y)$ *for all* ordinals $\beta' < \beta$.

Proof (i) We want to show that L is a countable union of scattered orders, so it suffices to show that L_n is scattered $\forall n \in \omega$. We do this by induction on n. L_0 is scattered since $\sigma_{\alpha\beta} \in \mathcal{S}_2$ (immediate from definitions).

For $n > 0$, observe that $\text{tp}(L_n) = \sum_{x \in L_{n-1}} \phi_x$ where:

$$\phi_x = \begin{cases} 1 + \sigma_{\alpha\beta}, & \text{if } \exists y \text{ such that } (x, y) \text{ is empty} \\ 1 & \text{otherwise} \end{cases}$$

By the inductive hypothesis L_{n-1} is scattered, so by Lemma 3.1 (ii), L_n is scattered.

(ii) We prove that $\beta \not\leq \eta_{\alpha\beta}$; the α^* case is symmetric. Recall that β is regular since $\langle \alpha, \beta \rangle$ is admissable.

First, we show by induction that $\beta \not\leq \text{tp}(L_n)$ for all $n \in \omega$. For $n = 0$, we have $\beta \not\leq \sigma_{\alpha\beta}$ by Lemma 3.1 (iii) applied to the definition of $\sigma_{\alpha\beta}$. For $n > 0$, use the sum in part (i): $\text{tp}(L) = \sum_{x \in L_{n-1}} \phi_x$. From the $n = 0$ case, $\beta \not\leq \phi_x$ for all x, and by the inductive hypothesis $\beta \not\leq L_{n-1}$. Hence, by Lemma 3.1 (iii), $\beta \not\leq L_n$.

Since β is uncountable, $\omega < \beta$, hence we can apply Lemma 3.1 (iv) to conclude that $\beta \not\leq \text{tp}\big(\bigcup_{n \in \omega} L_n\big) = \eta_{\alpha\beta}$.

(iii) Again, we will only show this for β, with the α^* case being symmetric. Let $x < y \in L$. Then $x, y \in L_n$ for some n. Since L_n is scattered (from part (i)), there will be $x', y' \in L_n$ s.t. $x < x' < y' < y$ and (x', y') is empty in L_n. But then, by definition, $\sigma_{\alpha\beta} \subset (x', y')$ in L_{n+1}, so $\sigma_{\alpha\beta} \subset (x, y)$ in L. Hence, and by the definition of $\sigma_{\alpha\beta}$, we get that all *cardinals* less than β are embeddable in (x, y) for any $(x, y) \in L$.

To show that $\beta' \leq (x, y)$ for all $x < y \in L$ and for all *ordinals* less than β, we use ordinal induction. So let $\beta' < \beta$ and by the induction hypothesis assume that $\gamma \leq (x, y)$ $\forall x < y \in L$ and $\forall \gamma < \beta'$.

So pick any $x < y \in L$. Since cofinalities are cardinals, $\text{cf}(\beta') \leq (x, y)$; let $f : \text{cf}(\beta') \to (x, y)$ be an embedding. If $\beta' = \text{cf}(\beta')$ then we would be done, so assume that $\text{cf}(\beta') < \beta'$. Then $\beta' = \sum_{\delta \in \text{cf}(\beta')} \gamma_\delta$, where for all $\delta, \gamma_\delta < \beta'$. By the inductive hypothesis, γ_δ can be embedded into any interval in L, so in particular, it can be embedded into $(f(\delta), f(\delta^+))$; let g_δ be such an embedding. But then 'gluing together' these g_δ gives an embedding of β' into (x, y), as required.

$\square$

Lemma 3.4. *Let L and M be linear orders. If L satisfies Lemma 3.3 (iii) and if tp(M) satisfies Lemma 3.3 (i), (ii), then $M \leq L$.*

Proof First, we claim that if M' is a scattered ordering satisfying Lemma 3.3 (ii), then there is an embedding $f : M' \to L$ s.t. for all Dedekind cuts (M'_1, M'_2) of M, there exists an interval $(x_1, x_2) \subset L$, s.t.

$$y_1 \in M'_1, y_2 \in M'_2, x \in (x_1, x_2) \Rightarrow f(y_1) < x < f(y_2) \quad (*)$$

We prove the claim by induction on the recursive hierarchy of the scattered types (Thm 3.2). The claim is trivial for $\mathcal{S}_0$. For the induction step, we must first show the claim holds for well-orders less than β; by a symetrical argument, the claim holds for converse well-orders less than α^*.

So let $\gamma < \beta$ be a well-order. Since L satisfies (iii), there exists an embedding $g : \gamma \to L$. The only way this g cannot satisfy the claim is if there is a limit ordinal $\delta \in \gamma$ such that $\sup\{g(\zeta) : \zeta < \delta\} = g(\delta)$; in this case there would be no interval (x_1, x_2) satisfying $(*)$ for the Dedekind cut $(\{\zeta < \delta\}, \{\zeta \geq \delta\})$. To fix this, you consider $g' : \gamma \to L$ where $g'(\delta) = g(\delta + 1)$. Then, $\sup\{g'(\zeta) : \zeta < \delta\} = g(\delta) < g(\delta + 1) = g'(\delta)$. Hence, the interval $(g(\delta), g(\delta + 1))$ satisfies $(*)$ for the cut $(\{\zeta < \delta\}, \{\zeta \geq \delta\})$, and we have got an appropriate embedding.

(There is a very minor point, which is that if γ is a successor, then $\delta + 1$ can equal γ, so $g'(\delta)$ would not be defined. This is easily fixed though, by originally considering an embedding $g : (\gamma + 1) \to L$).

To complete the claim, let M' be a scattered type satisfying (ii). Then (by Thm 3.2), $M' = \sum_{\delta \in \gamma} M'_\delta$, where $\gamma < \beta$ (the case $\gamma < \alpha^*$ is symmetric). To embed M' into L as per the claim, you first embed γ into L, using g, say. Then (by the induction hypothesis and noting that any interval of L satisfies (iii)) embed M'_δ into $(g(\delta), g(\delta + 1))$, using f_δ, say. Then finally, 'glue together' the f_δ to get the required $f : M' \to L$. (C.f. proof of Lemma 3.3 (iii)). This completes the proof of the claim.

As $\text{tp}(M) \in \mathcal{M}$, $M = \bigcup_{n\in\omega} M_n$ where M_n are scattered and WLOG, the M_n are pairwise disjoint. Let $f_0 : M_0 \to L$ be an embedding as per the claim; for (M', M'') a Dedekind cut of M_0, let $F((M', M''))$ denote an interval in L satisfying $(*)$.

We extend f_0 to $f_1 : M_0 \cup M_1 \to L$ so that f_1 also satisfies the conditions of the claim. To do this, first observe that we can partition M_1 into intervals M_δ so that for each δ, there is a unique Dedekind cut (M'_δ, M''_δ) of M_0 such that $y \in M_\delta, y' \in M'_\delta$ and $y'' \in M''_\delta \Rightarrow y' < y < y''$. But then (noting that any interval of L also satisfies Lemma 3.3 (iii)), we can appropriately embed each M_δ into $F((M', M''))$, thus giving us the extension f_1.

We then extend f_1 to $f_2 : M_0 \cup M_1 \cup M_2 \to L$ so that f_2 satisfies the conditions of the claim, in the same manner. We continue this process (i.e. use induction), thus giving us an embedding $f : M \to L$, as required. $\square$

Corollary 3.5. $\psi \leq \eta_{\alpha\beta} \iff \psi \in \mathcal{M}$, *$\beta \not\leq \psi$, and $\alpha^* \not\leq \psi$.*

Proof $\Rightarrow$: Follows from Lemma 3.3 (i) and (ii).
$\Leftarrow$: Follows from Lemma 3.3 (iii) and Lemma 3.4. $\square$

Theorem 3.6. *(Uniqueness of $\eta_{\alpha\beta}$) If M is a linear order of type $\psi \neq 0, 1$ and if there exists $\langle \alpha, \beta \rangle$ such that M and ψ satisfy Lemma 3.3 (i), (ii) and (iii) (in place of L and $\eta_{\alpha\beta}$), then $\langle \alpha, \beta \rangle$ is admissable and $\psi \equiv \eta_{\alpha\beta}$.*

Proof Since $\psi \in \mathcal{M}$, write $M = \bigcup_{n\in\omega} M_n$ where M_n are scattered.

β (and by symmetry, α) is uncountable: We will show that $\omega \leq M$, which implies, by (ii), that $\beta > \omega$ i.e. β is uncountable. By assumption $|M| > 1$, so by (ii), $\beta > 2$. Hence, by (i), M is dense. Now let $x < y \in M$. We construct an infinite increasing sequence in M (which essentially *is* an embedding of ω) by letting $x_0 = x$, then by density let $x_1 \in (x_0, y)$, then by density let $x_2 \in (x_1, y)$, and so on.

β (and by symmetry, α) is regular: If not, then like the proof of Lemma 3.3 (iii), we can write β

as a cf(β) sum of smaller ordinals and hence, by using (iii),we can embed β into M, contradicting (ii).

$\max\{\alpha, \beta\}$ is a successor: Suppose it is a limit cardinal and, WLOG, suppose β is the maximum, the other case being symmetric. By (iii), all cardinals less than β can be embedded in M. Since β is a limit cardinal, this means $|M| \geq \beta$. As β is regular and uncountable, we can apply Lemma 3.1 (iv) to conclude that $|M_n| \geq \beta$ for some n. Then by Lemma 3.1 (v), β or $\beta^* \leq M_n$, so β or $\beta^* \leq M$, contradicting (ii) (since $\beta = \max\{\alpha, \beta\}$, $\beta^* \leq M \Rightarrow \alpha^* \leq M$).

$\psi \equiv \eta_{\alpha\beta}$: Immediate from Lemma 3.4. □

Corollary: $(\eta_{\alpha\beta})^2 \equiv \eta_{\alpha\beta}$ and for any $x < y \in L$ (where $\mathrm{tp}(L) = \eta_{\alpha\beta}$), $\mathrm{tp}((x, y)) \equiv \eta_{\alpha\beta}$.

Definition For $\langle\alpha, \beta\rangle$ admissable, $\mathcal{D}_{\alpha\beta} =_{def} \{\phi \in \mathcal{M} : \phi < \eta_{\alpha\beta}\}$.

Lemma 3.7. *A $\mathcal{D}_{\alpha\beta}$ sum of members of $\mathcal{D}_{\alpha\beta}$ is in $\mathcal{D}_{\alpha\beta}$.*

Proof Let $\mathrm{tp}(M) \in \mathcal{D}_{\alpha\beta}$ and for each $x \in M$, let $\phi_x \in \mathcal{D}_{\alpha\beta}$. Then we want to show that $\sum_{x\in M} \phi_x \in \mathcal{D}_{\alpha\beta}$. Clearly, $\sum_{x\in M} \phi_x \leq \sum_{y\in N} \eta_{\alpha\beta}$, where $\mathrm{tp}(N) = \eta_{\alpha\beta}$. But $\sum_{y\in N} \eta_{\alpha\beta} = (\eta_{\alpha\beta})^2 \equiv \eta_{\alpha\beta}$, so we get $\sum_{x\in M} \phi_x \leq \eta_{\alpha\beta}$. If $\eta_{\alpha\beta} \leq \sum_{x\in M} \phi_x$, then, just like in Lemma 3.1 (ii), we must have $\eta_{\alpha\beta} \leq M$ or $\eta_{\alpha\beta} \leq \phi_x$ for some x, but neither of these is possible. Hence, $\eta_{\alpha\beta} \not\leq \sum_{x\in M} \phi_x$, and so $\sum_{x\in M} \phi_x < \eta_{\alpha\beta}$, as required. □

Theorem 3.8. $\mathcal{D}_{\alpha\beta} = \bigcup_{\gamma\in On}(\mathcal{D}_{\alpha\beta})_\gamma$, *where* $(\mathcal{D}_{\alpha\beta})_\gamma$ *is defined recursively:* $(\mathcal{D}_{\alpha\beta})_0 = \{0, 1\}$ *and for* $\gamma > 0, \phi \in (\mathcal{D}_{\alpha\beta})_\gamma \iff_{def}$ ϕ *is an* α' *sum for* $\alpha' < \alpha^*$*, or, a* β' *sum for* $\beta' < \beta$*, or, an* $\eta_{\alpha'\beta'}$ *sum for* $\langle\alpha', \beta'\rangle < \langle\alpha, \beta\rangle$*, of elements of* $\bigcup_{\delta<\gamma}(\mathcal{D}_{\alpha\beta})_\delta$.

Remark: This is the recursive characterisation of $\mathcal{M}$ that has previously been referred to.

Remark: This result is due to Galvin (unpublished), and was communicated to Laver in writing.

Remark: A lot of the proof is identical to the proof of Thm 3.2, hence, I will skip over details which add nothing new.

Proof Let $\mathcal{C}_{\alpha\beta} = \bigcup_{\gamma\in\mathrm{On}}(\mathcal{D}_{\alpha\beta})_\gamma$. We will show that $\mathcal{C}_{\alpha\beta} \subseteq \mathcal{D}_{\alpha\beta}$ and $\mathcal{D}_{\alpha\beta} \subseteq \mathcal{C}_{\alpha\beta}$.

$\mathcal{C}_{\alpha\beta} \subseteq \mathcal{D}_{\alpha\beta}$: By Lemma 3.3 and 3.5, $\alpha' \in \mathcal{D}_{\alpha\beta}$ for all $\alpha' < \alpha^*$, $\beta' \in \mathcal{D}_{\alpha\beta}$ for all $\beta' < \beta$ and $\eta_{\alpha'\beta'} \in \mathcal{D}_{\alpha\beta}$ for all $\langle \alpha', \beta' \rangle < \langle \alpha, \beta \rangle$. Hence, we use ordinal induction and Lemma 3.7 to conclude that $\mathcal{C}_{\alpha\beta} \subseteq \mathcal{D}_{\alpha\beta}$ (c.f Thm 3.2).

$\mathcal{D}_{\alpha\beta} \subseteq \mathcal{C}_{\alpha\beta}$. First, a $\mathcal{C}_{\alpha\beta}$ sum of elements of $\mathcal{C}_{\alpha\beta}$ is in $\mathcal{C}_{\alpha\beta}$ (c.f. Thm 3.2). Then let L be a linear order with $\mathrm{tp}(L) \in \mathcal{D}_{\alpha\beta}$. Define a relation $\sim$ on L: let $x \sim y$ iff $x = y$, or, $x < y$ and $\mathrm{tp}((x, y)) \in \mathcal{C}_{\alpha\beta}$, or, $y < x$ and $x \sim y$. $\sim$ is an equivalence relation which partitions L into intervals. If $|x|$ is an equivalence class of $\sim$, then $\mathrm{tp}(|x|) \in T$ (c.f. Thm 3.2).

I claim that there is in fact only one equivalence class, L, which gives the theorem. Suppose the claim is false. Let L' be the set of equivalence classes of L, ordered by $|x| \leq |y|$ iff $x \leq y$. For any distinct $|x|, |y| \in L'$, $\mathrm{tp}((|x|, |y|)) \notin \mathcal{C}_{\alpha\beta}$ $(*)$. Otherwise $\mathrm{tp}(x, y)$ would be a $\mathcal{C}_{\alpha\beta}$ sum of elements of $\mathcal{C}_{\alpha\beta}$, which, contradicts $x \not\sim y$.

Now, since $\mathrm{tp}(L') \in \mathcal{D}_{\alpha\beta}$, and by Thm 3.6, there exists an interval $(|x_0|, |y_0|)$ into which some $\beta' < \beta$ or some $\alpha' < \alpha^*$ cannot be embedded. Suppose it is the first (with the α^* case being symmetric) and choose β' to be minimal, so that every $\beta'' < \beta'$ can be embedded in every interval of L'. Now let $\alpha' \leq \alpha$ be the least γ such that there exists a subinterval $(|x_1|, |y_1|) \subseteq (|x_0|, |y_0|)$ into which $\gamma*$ cannot be embedded. So we have:

(i) $\mathrm{tp}((|x_1|, |y_1|)) \in \mathcal{M}$,

(ii) $\beta' \not\leq (|x_1|, |y_1|)$ and $\alpha'^* \not\leq (|x_1|, |y_1|)$, and

(iii) For all $|x_2| < |y_2| \in (|x_1|, |y_1|)$, $\beta'' \leq (|x_2|, |y_2|)$ for all $\beta'' < \beta'$ and $\alpha'' \leq (|x_2|, |y_2|)$ for all $\alpha'' < \alpha'^*$.

Hence, by Thm 3.6, $\mathrm{tp}((|x_1|, |y_1|)) = \eta_{\alpha'\beta'}$, which is certainly in $\mathcal{C}_{\alpha\beta}$, contradicting $(*)$. $\square$

4 The Main Theorem

The purpose of this chapter is prove the main theorem of the essay. In order to state the theorem, we must first introduce the basic notions regarding Q-orders. These are all straightforward extensions of definitions in Chapter 3.

Definition Let Q be a QO. A *Q-labelled linear order*, abbreviated to *Q-order*, is a pair $\langle L, l\rangle$ where L is a linear order and l is a function from L to Q; intuitively, you are labelling each element of L with an element from Q. Two Q-orders $\langle L, l\rangle$ and $\langle M, l'\rangle$ are isomorphic iff$_{def}$ there is a bijective, order-preserving $f : L \to M$ such that $\forall x \in L, l(x) = l'(f(x))$. The *$Q$-type* of $\langle L, l\rangle$, written $\text{tp}(\langle L, l\rangle)$, is the class of Q-orders isomorphic to $\langle L, l\rangle$.

In this essay, we will use Φ, Ψ, Θ and χ to range over Q-types.

Definition $f : L \to M$ *embeds* $\langle L, l\rangle$ into $\langle M, l'\rangle$ iff$_{def}$ f is injective, order preserving, and $\forall x \in L, l(x) \leq_Q l'(f(x))$. $\Phi \leq \Psi$ iff$_{def}$ there exists $\langle L, l\rangle, \langle M, l'\rangle$ and $f : L \to M$ such that $\text{tp}(\langle L, l\rangle) = \Phi$, $\text{tp}(\langle M, l'\rangle) = \Psi$ and f embeds $\langle L, l\rangle$ into $\langle M, l'\rangle$.

Definition The *ordered sum* $\sum_{x\in M}\langle L_x, l_x\rangle$ is the Q-order obtained by replacing each $x \in M$ by $\langle L_x, l_x\rangle$. Similarly, $\sum_{x\in M}\Phi_x$ is the type of $\sum_{x\in M}\langle L_x, l_x\rangle$ where $\text{tp}(\langle L_x, l_x\rangle) = \Phi_x$ for all $x \in M$.

Definition If $\Phi = \text{tp}(\langle L, l\rangle)$, the *base* of Φ is $\text{tp}(L)$. For $\phi \in \text{LO}$, $Q^{\phi}(Q^{\leq\phi}, Q^{\equiv\phi})$ is$_{def}$ the set of Q-types Φ such that $\text{bs}(\Phi) = \phi$ $(\leq \phi, \equiv \phi)$. If $\mathcal{R} \subseteq \text{LO}$, then $Q^{\mathcal{R}}$ is the set of Q-types Φ such that $\text{bs}(\Phi) \in \mathcal{R}$.

Remark: Where no confusion arises, we sometimes abuse notation by writing Φ (or ϕ) when we really mean a Q-order of type Φ (or linear order of type ϕ).

Now we are ready to state the main theorem of the essay:

Theorem Q BQO implies $Q^{\mathcal{M}}$ BQO.

Before continuing, we provide a sketch of the structure of the proof. The proof is given by the following chain of implications:

$$Q \text{ BQO} \overset{\text{(i)}}{\Longrightarrow} Q^+\text{BQO} \overset{\text{(ii)}}{\Longrightarrow} \tau_{Q^+} \text{ BQO} \overset{\text{(iii)}}{\Longrightarrow} \mathcal{H}(Q) \text{ BQO} \overset{\text{(iv)}}{\Longrightarrow} \mathcal{H}(Q)^{<\omega} \text{ BQO} \overset{\text{(v)}}{\Longrightarrow} Q^{\mathcal{M}} \text{ BQO}$$

(where Q^+ and $\mathcal{H}(Q)$ are to be defined). A few anticipatory remarks:

1) We already have (ii) and (iv), by Cor 2.13 and Thm 2.11 (v).

2) (i) will be immediate from the definition of Q^+ and Thm 2.11 (ii) (iii) and (iv).

3) (iii) is a fairly simple induction, and makes sense of the definition of Q^+.

4) The proof of (v) is where we require the use of the recursive characterisation of $\mathcal{M}$ and of Q-labelled LOs, (as opposed to non-labelled LOs).

In order to construct $\mathcal{H}(Q)$, we first need to introduce a couple of constructions. We also prove a simple lemma for each construction; these lemmas are central in proving (iii).

Definition Let $\mathcal{U}$ be a set of Q-types and κ an infinite cardinal. Then Φ is an *unbounded* $(\mathcal{U}, \kappa)$-*sum*, abbreviated to $(\mathcal{U}, \kappa)$-sum, iff$_{def}$ Φ can be written as $\sum_{x \in L} \Phi_x$ where $\text{tp}(L) = \kappa$, $\{\Phi_x : x \in M\} = \mathcal{U}$ and $\forall x \in L$, $\exists Y \subseteq L$ s.t. $|Y| = \kappa$ and $y \in Y \Rightarrow \Phi_x \leq \Phi_y$. We refer to this last condition as unboundedness. You get a $(\mathcal{U}, \kappa^*)$-sum by replacing κ with κ^* in the definition.

Lemma 4.1. *Let $\delta \in RC$, $\kappa \leq \delta$ be an infinite cardinal and $\mathcal{U}, \mathcal{V}$ be sets of Q-types such that $\forall \Theta \in \mathcal{U}\ \exists \chi \in \mathcal{V}$ s.t. $\Theta \leq \chi$. Then, if Φ is a $(\mathcal{U}, \kappa)$-sum and Ψ is a $(\mathcal{V}, \delta)$-sum (or, if Φ is a $(\mathcal{U}, \kappa^*)$-sum and Ψ is a $(\mathcal{V}, \delta^*)$-sum), then $\Phi \leq \Psi$.*

Proof Write Φ and Ψ as $(\mathcal{U}, \kappa)$ and $(\mathcal{V}, \delta)$ sums (the κ^*, δ^* case is symmetric): $\Phi = \sum_{x \in L} \Phi_x$ and $\Psi = \sum_{y \in M} \Psi_y$. We define an embedding $f : \Phi \to \Psi$ by induction on L (noting that $\text{tp}(L) = \kappa$); so suppose we have defined f on an initial segment $\sum_{x < x_0} \Phi_x$ of Φ to an initial segment $\sum_{y < y_0} \Psi_y$ of Ψ. We want to extend f so it is defined on $\sum_{x \leq x_0} \Phi_x$ and to some initial segment of Ψ.

To get the extension, it suffices to find $y \in M$ s.t. $y \geq y_0$ and $\Phi_{x_0} \leq \Psi_y$. By definition of an unbounded sum, $\Phi_{x_0} \in \mathcal{U}$. By assumption, $\exists \chi \in \mathcal{V}$ s.t. $\Phi_{x_0} \leq \chi$. Again by the definition of an unbounded sum, there exists some $y_1 \in M$ s.t. $\Psi_{y_1} = \chi$. By unboundedness, there exists δ many $y \in M$ s.t. $\Psi_y \geq \Psi_{y_1}$. Hence, and since $\kappa \leq \delta$ and δ is regular, there exists $y > y_0$ s.t. $\Psi_y \geq \Psi_{y_1} = \chi \geq \Phi_{x_0}$, as required.

Note that for $\gamma < \kappa$, the limit of a γ sequence of initial segments of M is itself an initial segment of M, because $\kappa \leq \delta$ and δ is regular. Hence, the induction does succeed in defining the embedding on the whole domain. $\square$

Definition Let $\mathcal{U}$ be a set of Q-types and let Φ be a $\mathcal{U}$-type, i.e. $\Phi = \text{tp}(\langle L, l\rangle)$ where L is a linear order and $l : L \to \mathcal{U}$. Then $\overline{\Phi} =_{def} \sum_{x \in L} l(x)$.

Remark: We regard $\mathcal{U}$ as a subclass of the class of Q-types, so it is quasi-ordered by embeddability. Therefore, '$\mathcal{U}$-type' is a well-defined concept.

Remark: Observe that if $\Phi \leq \Psi$ are $\mathcal{U}$-types, then $\overline{\Phi} \leq \overline{\Psi}$.

Definition Let Q be a quasi-order and $\langle \alpha, \beta\rangle$ be admissable. A Q-type Φ is (Q, α, β)-*maximal* iff$_{def}$ $\Phi \in Q^{\equiv \eta_{\alpha\beta}}$ and whenever $\Psi \in Q^{\leq \eta_{\alpha\beta}}$ we have $\Psi \leq \Phi$.

Definition Let $\mathcal{U}$ be a set of Q-types (recall that $\mathcal{U}$ is a quasi-order). A Q-type Φ is a $(\mathcal{U}, \alpha, \beta)$ *maximal sum*, or a $(\mathcal{U}, \alpha, \beta)$-*sum* for short, iff$_{def}$ there exists a $\mathcal{U}$-type Ψ such that Ψ is $(\mathcal{U}, \alpha, \beta)$-maximal and $\Phi = \overline{\Psi}$.

Lemma 4.2. *Let $\mathcal{U}, \mathcal{V}$ be sets of Q-types such that $\forall \Theta \in \mathcal{U}\ \exists \chi \in \mathcal{V}$ s.t. $\Theta \leq \chi$. Also let $\langle \alpha, \beta\rangle \leq \langle \gamma, \delta\rangle$. Then Φ a $(\mathcal{U}, \alpha, \beta)$-sum and Ψ a $(\mathcal{V}, \gamma, \delta)$-sum implies $\Phi \leq \Psi$.*

Proof We have $\Phi = \sum_{x \in L} \Theta_x$ where $\text{tp}(L) \equiv \eta_{\alpha\beta}$ and $\Theta_x \in \mathcal{U}$ for all x. By assumption, $\exists \chi_x \in \mathcal{V}$ such that $\Theta_x \leq \chi_x$ for all x. Hence, $\Phi \leq \sum_{x \in L} \chi_x = \Psi_0$, say. Let $\Psi_0' \in \mathcal{V}^{\equiv \eta_{\alpha\beta}}$ be the type of $\langle L, l\rangle$, where $l(x) = \chi_x$ for all x, so that $\overline{\Psi_0'} = \Psi_0$.

Now, since Ψ is a $(\mathcal{V}, \gamma, \delta)$-sum, let Ψ' be $(\mathcal{V}, \gamma, \delta)$-maximal such that $\overline{\Psi'} = \Psi$. Since $\langle \alpha, \beta\rangle \leq \langle \gamma, \delta\rangle$, $\Psi_0' \in \mathcal{V}^{\leq \eta_{\gamma\delta}}$, so by the definition of $(\mathcal{V}, \alpha, \beta)$-maximal, $\Psi_0' \leq \Psi'$. Hence,

$$\Phi \leq \Psi_0 = \overline{\Psi_0'} \leq \overline{\Psi'} = \Psi, \text{ as required.}$$

$\square$

The last precursor to the definition of $\mathcal{H}(Q)$:

Definition We write 0 for the Q-type with base 0. We write 1_q for $\text{tp}(\langle \{x\}, l\rangle)$ where $l(x) = q$.

Definition $\mathcal{H}(Q) = \bigcup_{\alpha \in On} \mathcal{H}_\alpha(Q)$, where $\mathcal{H}_\alpha(Q)$ is defined recursively by:

$\mathcal{H}_0(Q) = \{0\} \cup \{1_q : q \in Q\}$,

$$\mathcal{H}_\alpha(Q) = \Big\{\Phi \in Q^{\mathcal{M}} : \exists \mathcal{U} \subseteq \bigcup_{\beta<\alpha} \mathcal{H}_\beta(Q) \text{ s.t. } \Phi \text{ is a } (\mathcal{U}, \kappa) \text{ or } (\mathcal{U}, \kappa^*) \text{ sum for some } \kappa \in \text{RC},$$
$$\text{or, } \Phi \text{ is a } (\mathcal{U}, \gamma, \delta) \text{ sum for admissable } \langle \gamma, \delta \rangle\Big\}, \text{ for } \alpha > 0.$$

Remark: Unlike $\mathcal{S}$ or $\mathcal{M}$, $\mathcal{H}(Q)$ is not 'closed downwards': $\exists \Phi \leq \Psi \in \mathcal{M}$ s.t. $\Psi \in H$ but $\Phi \notin H$. This is why bs(Φ) needs to only be *equivalent* to $\eta_{\alpha\beta}$ for Φ to be (Q, α, β)-maximal; if we forced bs(Φ) to be equal to $\eta_{\alpha\beta}$, then we miss those Φ whose base is equivalent but not equal to $\eta_{\alpha\beta}$.

One cannot be blamed if intuition is lacking for this beastly construction. Some ease may be brought by the fact that if Q is BQO, then $\mathcal{H}(Q)$ is the class of indecomposable elements of $Q^{\mathcal{M}}$, where Φ is indecomposable if $\Phi_1 + \Phi_2 = \Phi \Rightarrow \Phi \leq \Phi_1$ or $\Phi \leq \Phi_2$. This fact may also help understand why Thm 4.7 is true.

Next we define Q^+.

Definition Let $A = \{a_\kappa : \kappa \in \text{ RC}\}$, $B = \{b_\kappa : \kappa \in \text{ RC}\}$ and $C = \{c_{\alpha\beta} : \langle \alpha, \beta \rangle \text{ admissable}\}$, which are quasi-ordered as follows: $a_\kappa \leq a_{\kappa'} \Leftrightarrow \kappa \leq \kappa' \Leftrightarrow b_\kappa \leq b_{\kappa'}$ and $c_{\alpha\beta} \leq c_{\alpha'\beta'} \Leftrightarrow \langle \alpha, \beta \rangle \leq \langle \alpha', \beta' \rangle$. Then, Q^+ is the disjoint union of Q, A, B and C.

Remark: In the case that Q^+ is not disjoint with A, B or C (e.g. $a_\kappa \in Q$), one relabels the elements of Q (e.g. replace a_κ with d_κ) to ensure that we do indeed get a disjoint union.

Lemma 4.3. *Q BQO $\Rightarrow \tau_{Q^+}$.*

Proof By Thm 2.11 (ii) A and B are BQO, and, by Thm 2.11 (ii) and (iv) C is BQO. Then by Thm 2.11 (iii), Q^+ is BQO. Then by Cor 2.13, τ_{Q^+} is BQO. □

Before proving τ_{Q^+} BQO $\Rightarrow \mathcal{H}(Q)$ BQO, we first establish some notation. Recall that $\rho(T)$ is the root of the tree T, $S(x)$ is the set of immediate successors of x, and br(x) is the subtree rooted at x.

Definition Let $q \in Q$ and τ' be a subset (not a subclass!) of τ_Q. Then $[q; \tau'] =_{def} (T, l) \in \tau_Q$ where $l(\rho(T)) = q$, $\{\text{br}(x) : x \in S(\rho(T))\} = \tau'$, and for $x, y \in S(\rho(T))$, $\text{br}(x) = \text{br}(y) \Rightarrow x = y$.

Definition For $q \in Q, 1^q$ is$_{def}$ the one element tree labelled by q.

Lemma 4.4. τ_{Q^+} $BQO \Rightarrow \mathcal{H}(Q)$ BQO

Proof We start by definining a function $T : \mathcal{H}(Q) \to \tau_{Q^+}$, by induction on $\mathcal{H}(Q)$. For $\mathcal{H}_0(Q), T(0) =$ the empty tree, and $\forall q \in Q, T(1_q) = 1^q$. Now let $\Phi \in \mathcal{H}_\alpha(Q)$ for $\alpha > 0$. By definition, for some $\mathcal{U} \subseteq \bigcup_{\beta<\alpha} \mathcal{H}_\alpha(Q)$, either:

(i) Φ is a $(\mathcal{U}, \kappa)$-sum for some $\kappa \in$ RC; then we let $T(\Phi) = [a_\kappa; \{T(\Theta) : \Theta \in \mathcal{U}\}]$,

(ii) Φ is a $(\mathcal{U}, \kappa^*)$-sum for some $\kappa \in$ RC; then we let $T(\Phi) = [b_\kappa; \{T(\Theta) : \Theta \in \mathcal{U}\}]$, or,

(iii) Φ is a $(\mathcal{U}, \alpha, \beta)$-sum for some admisable $\langle \alpha, \beta \rangle$; then we let $T(\Phi) = [c_{\alpha\beta}; \{T(\Theta) : \Theta \in \mathcal{U}\}]$.

Claim: Let $\Phi \in \mathcal{H}_\alpha(Q)$ for some $\alpha, \Psi \in \mathcal{H}_\beta(Q)$ for some β, and $T(\Phi) \leq_m T(\Psi)$. Then, $\Phi \leq \Psi$.

Proof of claim: Let $(T_1, l_1) = T(\Phi)$ and $(T_2, l_2) = T(\Psi)$. We prove the claim by induction, so assume we have the result for all $\langle \alpha', \beta' \rangle < \langle \alpha, \beta \rangle$. and let $f : T(\Phi) \to T(\Psi)$ be an embedding. If T_1 is empty, then the claim is trivially true, so from now on assume T_1 non-empty.

First suppose that $f(\rho(T_1)) = x \neq \rho(T_2)$. Then f is an embedding from $T(\Phi)$ to br(x). But observe that br(x) must equal $T(\chi)$, for some $\chi \in \mathcal{H}_\gamma(Q)$, where $\gamma < \beta$ and $\chi \leq \Psi$. Then, by the induction hypothesis, we get $\Phi \leq \chi$, so, $\Phi \leq \Psi$.

Now suppose $f(\rho(T_1)) = \rho(T_2)$. We have four cases:

Case 1: $l(\rho(T_1)) = q$ for some $q \in Q$, and since f is an embedding, $l(\rho(T_2)) = q' \geq q$. Then, $T(\Phi) = 1^q, T(\Psi) = 1^{q'}$, so $\Phi = 1_q \leq 1_{q'} = \Psi$.

Case 2: $l(\rho(T_1)) = a_\kappa$ for some $\kappa \in$ RC, and so $l(\rho(T_2)) = \delta \in$ RC such that $\kappa \leq \delta$. By definition of T, we have: Φ is a $(\mathcal{U}, \kappa)$-sum, where $\mathcal{U} = \{\Theta : T(\Theta) = \text{br}(x) \text{ for some } x \in S(\rho(T_1))\}$, and,

Ψ is a $(\mathcal{V}, \delta)$-sum, where $\mathcal{V} = \{\chi : T(\chi) = \text{br}(y) \text{ for some } y \in S(\rho(T_2))\}$.

Since f is an embedding, $\forall x \in S(\rho(T_1))$, $\exists y \in S(\rho(T_2))$ such that br$(x) \leq$ br(y). Hence, and by the inductive hypothesis, $\forall \Theta \in \mathcal{U}\ \exists \chi \in \mathcal{V}$ s.t. $\Theta \leq \chi$. Then, by Lemma 4.1, $\Phi \leq \Psi$.

Case 3: $l(\rho(T_1)) = b_\kappa$ for some $\kappa \in$ RC. Identical to Case 2 but with $(\mathcal{U}, \kappa^*)$ and $(\mathcal{V}, \delta^*)$-sums.

Case 4: $l(\rho(T_1)) = c_{\alpha\beta}$ for some $\langle \alpha, \beta \rangle$ and so $l(\rho(T_2)) = c_{\gamma\delta}$ such that $\langle \alpha, \beta \rangle \leq \langle \gamma, \delta \rangle$. By definition of T, we have that Φ a $(\mathcal{U}, \alpha, \beta)$-sum and Ψ a $(\mathcal{V}, \gamma, \delta)$-sum where, just like in Case 2, $\forall \Theta \in \mathcal{U}\ \exists \chi \in \mathcal{V}$ s.t. $\Theta \leq \chi$. Then, by Lemma 4.2, $\Phi \leq \Psi$. This concludes the proof of the claim.

But then considering $f : T``(\mathcal{H}(Q)) \subset \tau_{Q+} \to \mathcal{H}(Q)$, where $f(T)$ is mapped to some Φ s.t. $T(\Phi) = T$, we can apply the homomorphism property to conclude that τ_{Q+} BQO $\Rightarrow \mathcal{H}(Q)$ BQO. □

Our final task is to prove that $\mathcal{H}(Q)$ BQO $\implies$ $Q^{\mathcal{M}}$ BQO. The following theorem is used in Thm 4.7 to reduce the case $\mathrm{bs}(\Phi) \equiv \eta_{\alpha\beta}$ to the case $\mathrm{bs}(\Phi) < \eta_{\alpha\beta}$. We note that Laver had only proved the result in the case $\alpha = \beta = \omega_1$; the full result was supplied to him by Galvin.

Theorem 4.5. *Let Q be WQO. Then $\Phi \in Q^{\equiv \eta_{\alpha\beta}} \Rightarrow \Phi$ is a $\mathcal{D}_{\alpha\beta}$-sum of 1_q's and (R, α', β')-maximal types, where $R \subseteq Q$ and $\langle \alpha', \beta' \rangle \leq \langle \alpha, \beta \rangle$.*

Remark: The structure of the proof is the same as that of Thm 3.2 and 3.8 (the recursive characterisations of $\mathcal{S}$ and $\mathcal{M}$), so as in Thm 3.8, unhelpful details will not be included.

Proof Say Φ is *nice* if it is a $\mathcal{D}_{\alpha\beta}$-sum of 1_q's and (R, α', β')-maximal types, where $R \subseteq Q$ and $\langle \alpha', \beta' \rangle \leq \langle \alpha, \beta \rangle$, so the theorem becomes $\Phi \in Q^{\equiv \eta_{\alpha\beta}} \Rightarrow \Phi$ is nice. We prove the theorem by induction on WQOs (Thm 2.3), so suppose that $\forall q \in Q$, the theorem holds for $Q_q (= \{r \in Q : p \not\leq r\})$.

Start by observing that a $\mathcal{D}_{\alpha\beta}$-sum of nice types is nice, by Lemma 3.7 (a $\mathcal{D}_{\alpha\beta}$-sum of types in $\mathcal{D}_{\alpha\beta}$ is in $\mathcal{D}_{\alpha\beta}$). Now let $\Phi \in Q^{\equiv \eta_{\alpha\beta}}$ and $\langle L, l \rangle$ have type Φ. Define a relation $\sim$ on L: let $x \sim y$ iff $x = y$, or, $x < y$ and $\mathrm{tp}((x, y))$ is nice, or, $y < x$ and $x \sim y$. $\sim$ is clearly an equivalence relation which partitions L into intervals.

If $|x|$ is an equivalence class of $\sim$, then $\mathrm{tp}(|x|)$ is nice. To show this, you write $|x|$ as a $\gamma^* + \delta$ sum of 1_q's and sub-intervals (c.f. Thm 3.2). Since $\mathrm{tp}(L) \equiv \eta_{\alpha\beta}$, and by Lem 3.4, $\gamma < \alpha$ and $\delta < \beta$, so in particular, $\gamma^* + \delta \in \mathcal{D}_{\alpha\beta}$. Hence, $\mathrm{tp}(|x|)$ is $\mathcal{D}_{\alpha\beta}$-sum of nice types, so $\mathrm{tp}(|x|)$ is nice.

Hence, if L itself is one equivalence class, then we are done. So from now assume this is not the case. Let L' be the set of equivalence classes of L, ordered by $|x| \leq |y|$ iff $x \leq y$. Now if $(|x|, |y|)$

is any interval in L', then $\mathrm{tp}((|x|,|y|)) \equiv \eta_{\alpha\beta}$. Otherwise, $\mathrm{tp}((|x|,|y|)) \in \mathcal{D}_{\alpha\beta}$, so then we could write (x, y) as a $\mathcal{D}_{\alpha\beta}$-sum of nice types, contradicting $x \not\sim y$.

Furthermore, $\forall q \in Q, \exists z \in L$ s.t. $|z| \in (|x|,|y|)$ and $l(z) \geq_Q q$ $(*)$. Otherwise, $\exists q \in Q$ such that $\{z \in L : |z| \in (|x|,|y|)\} = \{z \in L : |z| \in (|x|,|y|)$ and $l(z) \in Q_q\} = Z$, say. Then by the induction hypothesis, $\mathrm{tp}(\langle Z, l|_Z\rangle)$ is nice. But then, as before, we could write (x, y) as a $\mathcal{D}_{\alpha\beta}$-sum of nice types, contradicting $x \not\sim y$.

We now claim that Φ is in fact (Q, α, β)-maximal. First, fix some interval $(|x|,|y|) \subset L'$. Since $\mathrm{tp}((|x|,|y|)) = \eta_{\alpha\beta} \equiv \eta_{\alpha\beta}^2$, there exists disjoint subintervals $(|x_u|,|y_u|)$ for each $u \in \eta_{\alpha\beta}$ such that if $a < b, |z| \in (|x_a|,|y_a|), |z'| \in (|x_b|,|y_b|)$ then $z < z'$.

To prove the claim, suppose $\mathrm{tp}(\langle M, l'\rangle) \in Q^{\leq \eta_{\alpha\beta}}$; we want to find g which embeds this into $\langle L, l\rangle$. Letting $f : M \to \eta_{\alpha\beta}$ be an embedding, define $g : M \to L$ by sending $m \in M$ to some $z \in L$ s.t. $|z| \in (|x_{f(m)}|,|y_{f(m)}|)$ and $l'(z) \geq l(u)$; this is possible by $(*)$. Hence, $\mathrm{tp}(\langle M, l'\rangle) \leq \Phi$, and so Φ is (Q, α, β)-maximal, as claimed. But then Φ is nice, thus completing the proof.

(Strictly, the proof continues: Φ nice implies L is a $\sim$-equivalence class, contradicting our assumption that L is not an equivalence class. Hence, L is an equivalence class, which implies Φ is nice, as required). □

We also require the following straightforward lemma in Thm 4.7.

Lemma 4.6. *If $\chi \in \mathcal{H}_\gamma(\mathcal{H}(Q))$ (where $\mathcal{H}(Q)$ is a QO by regarding it as a subclass of $Q^{\mathcal{M}}$), then $\overline{\chi} \in \mathcal{H}(Q)$.*

Proof We prove this by induction on γ. If $\gamma = 0$, the result is trivial. Before continuing, observe that if $\chi = \sum_{l\in L} \chi_l$, then, $\overline{\chi} = \sum_{l\in L} \overline{\chi_l}$. Now, let $\gamma > 0$ and $\chi \in \mathcal{H}_\gamma(\mathcal{H}(Q))$.

If χ is a $(\mathcal{U}, \kappa)$-sum for some regular κ: Since $\Theta \leq \Theta' \Rightarrow \overline{\Theta} \leq \overline{\Theta'}$, it is clear that $\overline{\chi}$ is a $\big(\{\overline{\Theta} : \Theta \in \mathcal{U}\}, \kappa\big)$-sum. By the induction hypothesis, $\overline{\Theta} \in \mathcal{H}(Q)$ for all $\overline{\Theta} \in \mathcal{U}$, hence, $\overline{\chi} \in \mathcal{H}(Q)$.

If χ is a $(\mathcal{U}, \kappa^*)$-sum for some regular κ: A symmetric argument gives $\overline{\chi} \in \mathcal{H}(Q)$.

If χ is a $(\mathcal{U}, \alpha, \beta)$-sum for some admissable $\langle \alpha, \beta\rangle$: Then $\chi = \overline{\Phi}$ where $\Phi \in \mathcal{U}^{\equiv \eta_{\alpha\beta}}$ is $((\mathcal{U}, \alpha, \beta))$-

maximal, so:

$$\begin{aligned}\chi &= \sum_{x\in L} l(x), \text{ where } \mathrm{tp}(\langle L,l\rangle)=\Phi.\\ \Rightarrow\ \overline{\chi} &= \sum_{x\in L} l'(x), \text{ where for all } x,\ l'(x)=\overline{l(x)}\\ \Rightarrow\ \overline{\chi} &= \overline{\Phi'}, \text{ where } \Phi' = \mathrm{tp}(\langle L,l'\rangle).\end{aligned}$$

It is clear that $\Phi' \in \mathcal{U}'^{\equiv\eta_{\alpha\beta}}$, where $\mathcal{U}' = \{\overline{\Theta} : \Theta \in \mathcal{U}\}$, and further that Φ' is $(\mathcal{U}',\alpha,\beta)$-maximal. Hence, $\overline{\chi}$ is a $(\mathcal{U}',\alpha,\beta)$-sum. As before, the induction hypothesis implies that $\mathcal{U}' \subset \mathcal{H}(Q)$, and so $\overline{\chi} \in \mathcal{H}(Q)$. □

The next theorem is where the recursive characterisation of $\mathcal{M}$ is used and where the need for using Q-orders arises.

Theorem 4.7. *For all admissable $\langle\alpha,\beta\rangle$, if $\Phi \in Q^{\leq\eta_{\alpha\beta}}$ and Q BQO, then Φ is a finite sum of members of $\mathcal{H}(Q)$.*

Proof We prove this by induction, so assume the result holds for all admissable $\langle\alpha',\beta'\rangle < \langle\alpha,\beta\rangle$. We first show the theorem holds for $\Phi \in Q^{\mathcal{D}_{\alpha\beta}}$, using the recursive characterisation of $\mathcal{D}_{\alpha\beta}$, and then for $\Phi \in Q^{\equiv\eta_{\alpha\beta}}$, by using Thm 4.5 to reduce it to the $\Phi \in Q^{\mathcal{D}_{\alpha\beta}}$ case.

So our first task is to show that $\forall\gamma, \Phi \in Q^{(\mathcal{D}_{\alpha\beta})_\gamma}$ and Q BQO implies Φ is a finite sum of $\mathcal{H}(Q)$'s. We do this by induction on γ. The case $\gamma = 0$ is trivial, so let $\Phi \in Q^{(\mathcal{D}_{\alpha\beta})_\gamma}$ for $\gamma > 0$. By Thm 3.8, $\mathrm{bs}(\Phi)$ is either a β'-sum for $\beta' < \beta$, an α'-sum for $\alpha' < \alpha^*$, or, an $\eta_{\alpha'\beta'}$-sum for $\langle\alpha',\beta'\rangle < \langle\alpha,\beta\rangle$ of elements of $\bigcup_{\gamma'<\gamma}(\mathcal{D}_{\alpha\beta})_{\gamma'}$.

Case 1: $\mathrm{bs}(\Phi)$ is a β'-sum, so $\Phi = \sum_{x\in L}\Phi_x$ where $\mathrm{tp}(L) = \beta'$ and $\Phi_x \in \bigcup_{\gamma'<\gamma}(\mathcal{D}_{\alpha\beta})_{\gamma'}$. By the (second) induction hypothesis, each Φ_x is a finite sum of $\mathcal{H}(Q)'s$. Suppose Φ is not a finite sum of $\mathcal{H}(Q)$'s. WLOG, β' is minimal, so that if $\delta < \beta'$, any δ-sum of finite sums of $\mathcal{H}(Q)$'s is a finite sum of $\mathcal{H}(Q)$'s.

β' is infinite, as otherwise Φ would be a finite sum of $\mathcal{H}(Q)$'s. Further, $\mathrm{cf}(\beta') = \beta'$, so, $\beta' \in \mathrm{RC}$. To show this, you write $\Phi = \sum_{y\in M}\Phi_y$ where $\mathrm{tp}(M) = \mathrm{cf}(\beta')$ and Φ_y is a δ_y-sum for $\delta_y < \beta'$ of Φ_x's. By minimality of β', each Φ_y is a finite sum of $\mathcal{H}(Q)$'s, and by minimality of β' again, we conclude that $\mathrm{cf}(\beta') = \beta'$. Since for any $n_\delta \in \omega$, $\sum_{\delta<\beta'} n_\delta = \beta'$, we can write $\Phi = \sum_{x \in L} \Phi'_x$ where each $\Phi'_x \in \mathcal{H}(Q)$.

Next we claim that there exists $x' \in L$ such that $\sum_{x>x'} \Phi'_x$ is a $(\{\Phi'_x : x > x'\}, \beta')$-sum. Assume this is false. Since the other conditions are easily seen to be true, the claim can fail only if $\forall x' \in L, \sum_{x>x'} \Phi'_x$ is not unbounded, i.e. if $\forall x' \; \exists y > x'$ s.t. $\left|\{u \in L : y < u, \Phi'_y \leq \Phi'_u\}\right| < \beta'$. Hence, (and by regularity of β'), $\forall x' \; \exists y = y(x') > x' \; \exists z = z(y)$ such that $u \geq z(y) \Rightarrow \Phi'_{y(x')} \not\leq \Phi'_u$.

Now we can create a bad sequence $\Phi'_{y_0}, \Phi'_{y_1}, \ldots$ in $\mathcal{H}(Q)$: Pick any $x' \in L$. Let $y_0 = y(x'), y_1 = y(z(y_0)), y_2 = y(z(y_1))$, and so on. But Q BQO, so by Lemma 4.3, 4.4 and Thm 2.11 (i), $\mathcal{H}(Q)$ WQO, so no bad sequence can exist in $\mathcal{H}(Q)$. Contradiction. Hence, $\exists x' \in L$ s.t. $\sum_{x>x'} \Phi'_x$ is a $(\{\Phi'_x : x > x'\}, \beta')$-sum.

Hence, $\sum_{x>x'} \Phi'_x \in \mathcal{H}(Q)$, by the definition of $\mathcal{H}(Q)$ and the fact $\Phi'_x \in \mathcal{H}(Q)$ for all $x \in L$. So we have: $\Phi = \left(\sum_{x \leq x'} \Phi'_x\right) + \Psi$, some $\Psi \in \mathcal{H}(Q)$. But $\mathrm{tp}(\{x \in L : x \leq x'\}) < \beta'$, so by minimality of β', we conclude that Φ is a finite sum of $\mathcal{H}(Q)$'s. This contradicts our initial supposition, so Φ is indeed a finite sum of $\mathcal{H}(Q)$'s.

Case 2: $\mathrm{bs}(\Phi)$ is an α'-sum for $\alpha' < \alpha^*$. This case is symmetric to Case 1.

Case 3: $\mathrm{bs}(\Phi)$ is an $\eta_{\alpha'\beta'}$-sum for $\langle \alpha', \beta' \rangle < \langle \alpha, \beta \rangle$, so we have $\Phi = \sum_{x \in \eta_{\alpha'\beta'}} \Phi_x$, where each $\Phi_x = \Phi_{x,0} + \ldots + \Phi_{x,n_x}$ is a finite sum of $\mathcal{H}(Q)$'s. So then,

$$\begin{aligned}
\Phi &= \sum_{x \in \eta_{\alpha'\beta'}} \left(\Phi_{x,0} + \ldots + \Phi_{x,n_x}\right) \\
&= \sum_{x \in \eta_{\alpha'\beta'}} \left(\ldots + 0 + \ldots + \Phi_{x,0} + \ldots 0 \ldots + \Phi_{x,n_x} + \ldots + 0 + \ldots\right) \\
&= \sum_{x \in \eta_{\alpha'\beta'}} \left(\sum_{y \in \eta_{\alpha'\beta'}} l_x(y)\right), \text{where } l_x(y) \text{ is defined in the obvious way} \\
&= \sum_{(x,y) \in (\eta_{\alpha'\beta'})^2} l_x(y) \\
&= \sum_{x \in \eta_{\alpha'\beta'}} l'(x), \text{ for some } l'.
\end{aligned}$$

The last step is possible since we have $(\eta_{\alpha'\beta'})^2 \equiv \eta_{\alpha'\beta'}$. What has been achieved is a representation of Φ as $\overline{\chi}$ for some $\chi \in (\mathcal{H}(Q))^{\eta_{\alpha'\beta'}}$. Since $\mathcal{H}(Q)$ is BQO and $\langle \alpha', \beta' \rangle < \langle \alpha, \beta \rangle$, we can apply the (first) induction hypothesis to get $\chi = \chi_0 + \ldots + \chi_n$ where each $\chi_i \in \mathcal{H}(\mathcal{H}(Q))$. But then, $\Phi = \overline{\chi_0} + \ldots + \overline{\chi_n}$, and by Lemma 4.6, each $\overline{\chi_i} \in \mathcal{H}(Q)$, so Φ is a finite sum of $\mathcal{H}(Q)$'s. This completes the $\Phi \in Q^{\mathcal{D}_{\alpha\beta}}$ case.

Now let $\Phi \in Q^{\equiv\eta_{\alpha\beta}}$. By Thm 4.5, Φ is a $\mathcal{D}_{\alpha\beta}$-sum of 1_q's and (R, α', β')-maximal types, where $R \subseteq Q$ and $\langle\alpha', \beta'\rangle \leq \langle\alpha, \beta\rangle$. But observe that for $R \subseteq Q$, an (R, α', β')-maximal type is a $(\{1_q : q \in R\}, \alpha', \beta')$-sum. Hence, in particular, Φ is a $\mathcal{D}_{\alpha\beta}$-sum of $\mathcal{H}(Q)'s$, so $\Phi = \overline{\chi}$ for some $\chi \in (\mathcal{H}(Q))^{\mathcal{D}_{\alpha\beta}}$. Like in Case 3 above, we use the induction hypothesis and Lemma 4.6 to conclude that Φ is a finite sum of $\mathcal{H}(Q)$'s, thus concluding the proof. □

We can now prove the main theorem:

Theorem 4.8. *Q BQO $\implies$ $Q^{\mathcal{M}}$ BQO.*

Proof Let $\Phi \in Q^{\mathcal{M}}$. Then by Cor 3.5 $\Phi \in Q^{\mathcal{D}_{\alpha\beta}}$ for some (admissable) $\langle\alpha, \beta\rangle$. Then by Thm 4.7, Φ is a finite sum of $\mathcal{H}(Q)$'s. Hence, the function $f : \mathcal{H}(Q)^{<\omega} \to Q^{\mathcal{M}}$ sending $(\Phi_0, \ldots, \Phi_n)$ to $\Phi_0 + \ldots + \Phi_n$ is surjective; it is also clearly order-preserving. Since Q BQO, we apply Lemma 4.3, 4.4 and Thm 2.11 (v) to get $\mathcal{H}(Q)^{<\omega} BQO$, and then by the homomorphism property, we get $Q^{\mathcal{M}}$ BQO. □

Corollary 4.9. *$\mathcal{M}$ is WQO*

Proof Let $Q = \{x\}$ be the one element quasi-order, so $Q^{\mathcal{M}}$ and $\mathcal{M}$ are isomorphic. Since Q is trivially BQO, Thm 4.8 gives $Q^{\mathcal{M}} \simeq \mathcal{M}$ BQO, and hence M WQO. □

Remark: One may wonder if we could prove $\mathcal{M}$ is BQO by simply proving all the results of Chapter 4 for the particular case $Q = \{x\}$, thus greatly simplifying the proof. However, as alluded to previously[1], this would fail in Thm 4.7. The reason is that in the proof of Thm 4.7, you need to consider $\mathcal{H}(Q)$, so you need to have established the theorems of Chapter 4 for $\mathcal{H}(Q)$, not just Q.

[1] "The next theorem [Thm 4.7] is [...] where the need for using Q-orders arises."

5 Conclusion

Recall how the original purpose was to prove Fraïssé's conjecture: the class of scattered types is WQO. It is remarkable that in trying to do so, Laver achieved a much more general result, which at the same time (inadvertently?) generalises Nash-Williams' result that Q BQO $\Rightarrow Q^{\mathrm{On}}$ BQO. It is interesting to note that, in the paper, Laver says he originally proved Fraïssé's conjecture by induction on Hausdorff's characterisation of $\mathcal{S}$ (so presumably, he proved $QBQO \Rightarrow Q^{\mathcal{S}}BQO$), but that Galvin suggested generalising to $\mathcal{M}$ with the aid of his (Galvin's) work.

The paper's significance does not lie only in the fact that it has resolved Fraïssé's conjecture (contributing to the general goal of determining which naturally occuring quasi-orders are WQO), but also in its contribution to the understanding of $\mathcal{M}$.

We conclude by stating a few results that can be proved using Thm 4.5.

Definition Let Q be a QO. For $q \in Q$, $|q| = \{r \in Q : r \equiv p\}$, and $Q_{\equiv} = \{|q| : q \in Q\}$.

Theorem Suppose Q is BQO and $|Q_{\equiv}| \leq \kappa$. Then:
(i) If $\alpha \in$ On s.t. $\alpha < \kappa^+$, then $|(Q^\alpha)_{\equiv}| \leq \kappa$.
(ii) If $\langle \alpha, \beta \rangle$ is admissable and $\max\{\alpha, \beta\} \leq \kappa$, then $|(Q^{\leq \eta_{\alpha\beta}})_{\equiv}| \leq \kappa$.
From (i) it is immediate that if Q is BQO and $|Q_{\equiv}| \leq \kappa$, then $|\mathcal{P}(Q))_{\equiv}| \leq \kappa$.

Definition A type (or Q-type) ϕ is indecomposable iff$_{def}$ $\phi = \phi_1 + \phi_2$ then $\phi \leq \phi_i$ for some i.

Theorem If Q is BQO, then $\mathcal{H}(Q)$ is the class of indecomposable types of $\mathcal{M}$.

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